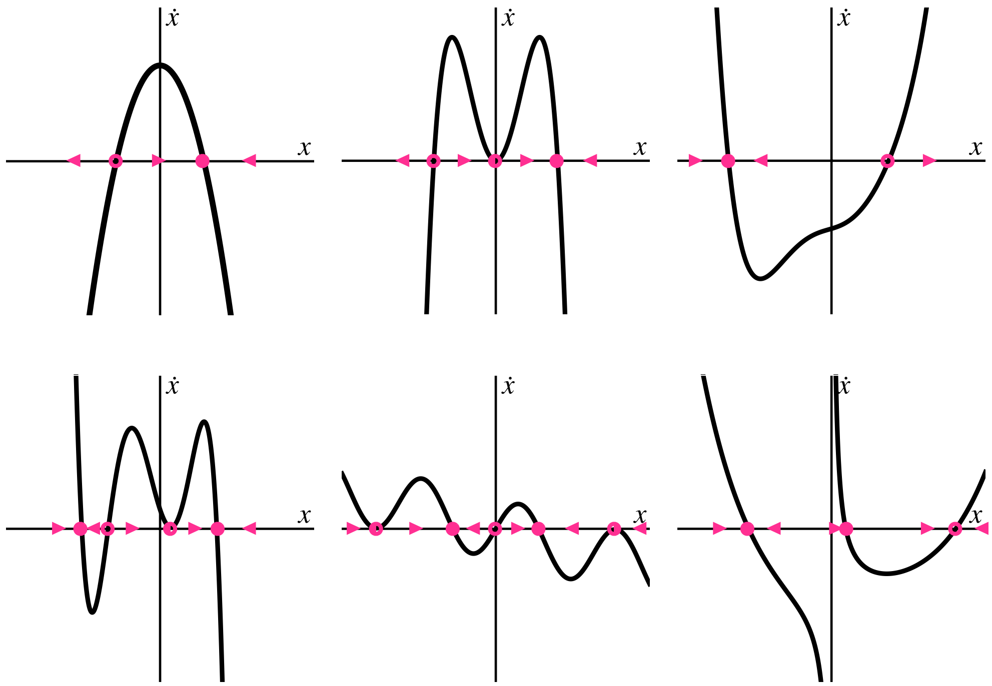
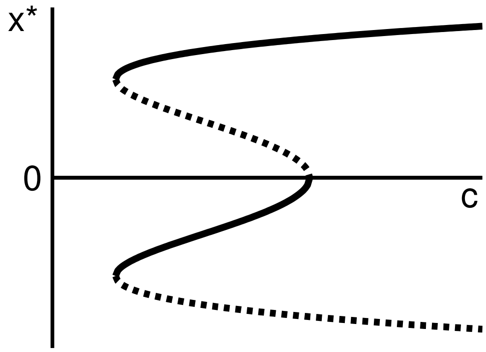
MOL 410/510: Introduction to Biological Dynamics Fall 2012
Problem Set #4, Nonlinear Dynamical Systems (due 10/19/2012)
6 **MUST DO** Questions, 1 **OPTIONAL** question

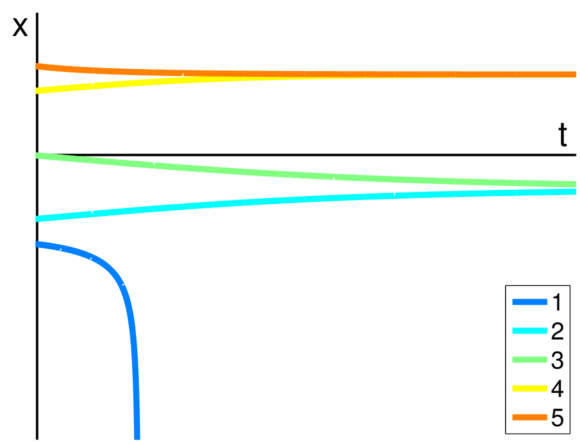
1. Solutions are drawn on the figure. Stable fixed points are closed circles, unstable fixed points are open circles, semi-stable points are shown as half closed circles with the closed side representing the side with flow into the fixed point and the open side representing the side with flow out of the fixed point. Arrows indicating flow are also shown.



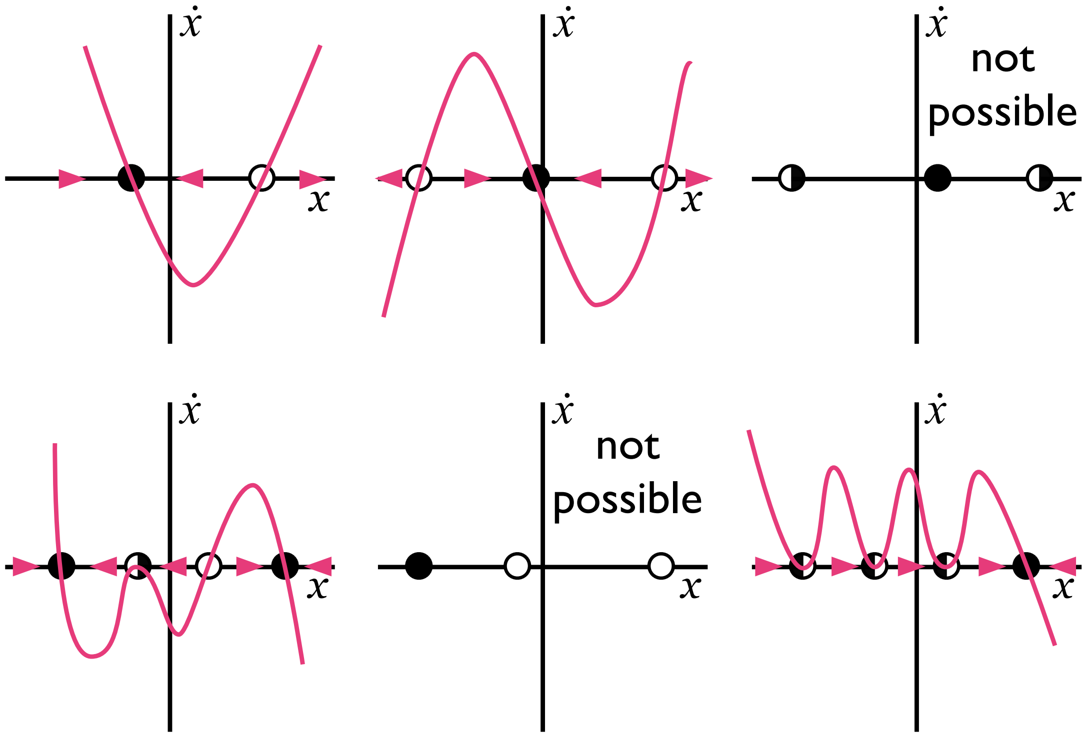
2. As the curve is shifted upwards due to increasing c, the fixed points will occur wherever the curve intersects with $\dot{x} = 0$. Fixed points furthest to the left (most negative value of x^*) will be unstable because the slopes are positive. The next fixed point to the right will be stable, the next unstable, and the last fixed point (furthest to the right, most positive value of x^* will be stable. Eventually, the two middle fixed points will disappear and only the end fixed points will remain.



3. Solutions are shown on the curve and color coded.

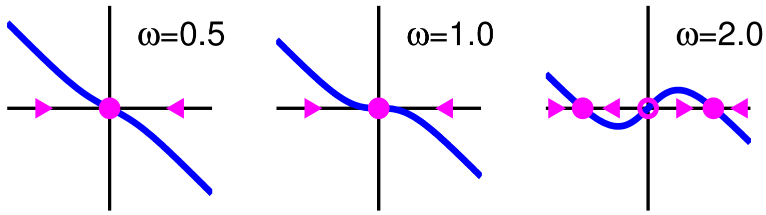


4. Solutions are shown on the figure.



5. **Autapse circuit**

   (a) The solution is shown in the figure, closed circles represent stable fixed points and open circles represent unstable fixed points.



   (b) We can approximate $\dot{x}$ about the fixed point $x^* = 0$ using

$$\dot{x}(t) \approx \dot{x(t)}\Big|_{x^*=0} + (x - x^*)\frac{d\dot{x}(t)}{dx}\Big|_{x^*=0} = (x - x^*)(-1 + w(1 - \tanh^2 x^*)) = x(w - 1)$$

   (c) Therefore, the value of w determines the stability of the fixed point: the fixed point is stable when $w < 1$ and unstable when $w > 1$. When $w = 0$, the system is at a critical point. Clearly, the flow is toward the fixed point overall, but linearly speaking, very close to the fixed point, there is no flow towards it. We call this critical slowing down.

   (d) Remember the Euler update equation approximates the dynamics as little pieces of line segments; the current value is the previous value plus an amount given by the slope of the dynamics ($\dot{x}$) at that point times the time step. Written as an equation:

$$x_{n+1} = x_n + \delta t\, \dot{x}(x_n) = x_n + \delta t(-x_n + w \tanh x_n)$$

   (e) See attached code for figure and simulation. Even though we are in a regime where there is still a single fixed point, the curve for when $w = 1$ approaches the fixed point at zero much slower than when $w = 0.25$. Again, this is an example of the critical slowing down that can happen at critical points. From the phase portrait it is obvious that when $w = 1$ the value of $\dot{x}$ is nearly zero for a wide range, this makes the dynamics much slower than when $w = 0.25$.

   (f) See attached code. For $\sigma = 0$, you should see that $x = 0$ for all time points. For $\sigma \neq 0$, x should either go up to the positive fixed point or go down to the negative fixed point. On average, it should go to either equally. However, with only 10 samples, you may get an asymmetry in your results.

6. **Two-dimensional circuit.**

(a) The nullclines are obtained by setting $\dot{y} = 0$ and $\dot{x} = 0$. Therefore, the nullclines are $y = \tanh^{-1}(-x/w)$ and $x = \tanh^{-1}(-y/w)$. To see the nullclines, refer to the code. For $w = 0.5$, there is one stable fixed point. For $w = 2$, there is one unstable fixed point and two stable fixed points. When $w = 0.5$, the lone fixed point represents both neurons firing at a medium rate. When $w = 2$, the unstable fixed point represents both neurons firing at a medium rate, and the two stable fixed points represent one neuron high and one neuron low.

(b) We can approximate $\dot{x}$ and $\dot{y}$ about a fixed point using

$$\dot{x} \approx (x - x^*)\frac{\partial \dot{x}}{\partial x}\bigg|_{x^*,y^*} + (y - y^*)\frac{\partial \dot{x}}{\partial y}\bigg|_{x^*,y^*} = -(x - x^*) - (y - y^*)w(1 - \tanh^2 y^*)$$

$$\dot{y} \approx (x - x^*)\frac{\partial \dot{y}}{\partial x}\bigg|_{x^*,y^*} + (y - y^*)\frac{\partial \dot{y}}{\partial y}\bigg|_{x^*,y^*} = -(x - x^*)w(1 - \tanh^2 x^*) - (y - y^*)$$

(c) Substituting $x^* = 0$ and $y^* = 0$ and turning the linearized equations into matrix form, we have

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -1 & -w \\ -w & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

The characteristic equation is $(-1 - \lambda)^2 - w^2 = 0$, therefore $\lambda = -1 \pm w$. The fixed point is stable so long as $|w| < 1$. Beyond that, it becomes a saddle point since one of the eigenvectors will be positive.

(d) See attached Matlab code.

(e) See attached Matlab code.