MOL 410/510: Biological Dynamics
Fall 2008
Problem Set #7
Solutions

1. (**5'**) The Gaussian distribution is described by the density function

$$G(x)=\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

The mean value of X is (**2'**) :

$$\overline{X}=\int_{-\infty}^{\infty}xG(x)dx=\int_{-\infty}^{\infty}x\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)dx$$

make substitution $y=(x-\mu)/\sqrt{2\sigma^2}$

$$\overline{X}=\int_{-\infty}^{\infty}\left(y\sqrt{2\sigma^2}+\mu\right)\frac{1}{\sqrt{\pi}}\exp\left(-y^2\right)dy=\frac{\sqrt{2\sigma^2}}{\sqrt{\pi}}\underbrace{\int_{-\infty}^{\infty}y\exp\left(-y^2\right)dy}_{=0\text{ since integrad is odd}}+\mu\int_{-\infty}^{\infty}\frac{1}{\sqrt{\pi}}\exp\left(-y^2\right)dy$$

$$\overline{X}=\mu$$

Likewise, the variance of X is given by (**3'**)

$$\operatorname{var}(X)=\int_{-\infty}^{\infty}\left(x-\mu\right)^2G(x)dx=\int_{-\infty}^{\infty}\left(x-\mu\right)^2\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)dx$$

make the substitution $y=(x-\mu)/\sqrt{2\sigma^2}$,

$$\operatorname{var}(X)=\int_{-\infty}^{\infty}\frac{\sqrt{2\sigma^2}}{\sqrt{\pi}}y^2\exp\left(-y^2\right)\sqrt{2\sigma^2}dy=\frac{2\sigma^2}{\sqrt{\pi}}\int_{-\infty}^{\infty}y^2\exp\left(-y^2\right)dy$$

To evaluate the final integral above, we can integrate by parts, using

$$\begin{aligned}u&=y & dv&=ye^{-y^2}dy\\ du&=dy & v&=-\frac{1}{2}e^{-y^2}\end{aligned}\qquad \int udv=uv-\int vdu$$

So $$\int_{-\infty}^{\infty}y^2\exp\left(-y^2\right)dy=\underbrace{y\left(-\frac{1}{2}\right)e^{-y^2}\Big|_{-\infty}^{\infty}}_{=0}+\frac{1}{2}\int_{-\infty}^{\infty}\exp\left(-y^2\right)dy=\frac{\sqrt{\pi}}{2}$$

Therefore,

$$\operatorname{var}(X)=\sigma^2.$$

2. (**5'**) Plugging in

$$f_1(y-x) = \frac{1}{\sigma_1\sqrt{2\pi}} e^{-(y-x)^2/2\sigma_1^2}$$

and

$$f_2(x) = \frac{1}{\sigma_2\sqrt{2\pi}} e^{-x^2/2\sigma_2^2}$$

gives the following integral

$$\begin{aligned}
P(y) &= \int_{-\infty}^{\infty} dx \frac{1}{\sigma_1\sqrt{2\pi}} e^{-(y-x)^2/2\sigma_1^2} \frac{1}{\sigma_2\sqrt{2\pi}} e^{-x^2/2\sigma_2^2} \\
&= \frac{1}{2\pi\sigma_1\sigma_2}\int_{-\infty}^{\infty} dx \exp[\frac{-(\sigma_1^2+\sigma_2^2)x^2 - \sigma_2^2 y^2 + 2\sigma_2^2 xy}{2\sigma_1^2\sigma_2^2}] \\
&= \frac{1}{2\pi\sigma_1\sigma_2}\int_{-\infty}^{\infty} dx \exp[\frac{-(\sigma_1^2+\sigma_2^2)}{2\sigma_1^2\sigma_2^2}(x^2 - \frac{2xy\sigma_2^2}{(\sigma_1^2+\sigma_2^2)} + \frac{y^2\sigma_2^2}{(\sigma_1^2+\sigma_2^2)})]
\end{aligned}$$

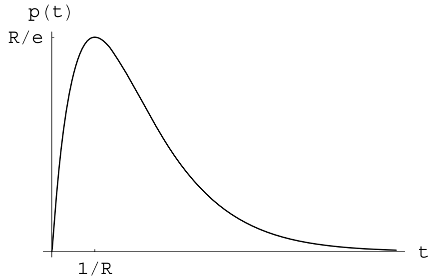
(Complete the square in the exponent)

$$\begin{aligned}
&= \frac{1}{2\pi\sigma_1\sigma_2}\int_{-\infty}^{\infty} dx \exp[\frac{-(\sigma_1^2+\sigma_2^2)}{2\sigma_1^2\sigma_2^2}((x - \frac{y\sigma_2^2}{(\sigma_1^2+\sigma_2^2)})^2 - \frac{y^2\sigma_2^4}{(\sigma_1^2+\sigma_2^2)^2} + \frac{y^2\sigma_2^2}{(\sigma_1^2+\sigma_2^2)})] \\
&= \frac{1}{2\pi\sigma_1\sigma_2}\int_{-\infty}^{\infty} dx \exp[\frac{-(\sigma_1^2+\sigma_2^2)}{2\sigma_1^2\sigma_2^2}((x - \frac{y\sigma_2^2}{(\sigma_1^2+\sigma_2^2)})^2 + \frac{y^2\sigma_1^2\sigma_2^2}{(\sigma_1^2+\sigma_2^2)^2})] \\
&= \frac{1}{2\pi\sigma_1\sigma_2} e^{-y^2/2(\sigma_1^2+\sigma_2^2)} \int_{-\infty}^{\infty} dx \exp[\frac{-(\sigma_1^2+\sigma_2^2)}{2\sigma_1\sigma_2}(x - \frac{y\sigma_2^2}{(\sigma_1^2+\sigma_2^2)})^2] \\
&= \frac{1}{2\pi\sigma_1\sigma_2} e^{-y^2/(\sigma_1^2+\sigma_2^2)} \sqrt{\frac{2\sigma_1^2\sigma_2^2\pi}{(\sigma_1^2+\sigma_2^2)}} \\
&= \frac{1}{\sqrt{2\pi(\sigma_1^2+\sigma_2^2)}} e^{-y^2/2(\sigma_1^2+\sigma_2^2)}
\end{aligned}$$

(Another Gaussian with variance $\sigma^2 = \sigma_1^2 + \sigma_2^2$)

3. **(8')**

(a) **(2')** The function $p(t) = R^2 te^{-Rt}$ has a maximum at $t = 1/R$, which we can determine by solving $p'(t) = 0$. The inflection point is at $t = 2/R$, which is calculated by solving $p''(t) = 0$. The curve of function $p(t)$ looks like the following:



(b) **(2')**

(c) **(2')** We compute the following integrals:

$$\int_0^\infty p(t)dt = \int_0^\infty R^2 te^{-Rt}dt$$

$$= \underbrace{R^2 t\left(-\frac{1}{R}e^{-Rt}\right)\Bigg|_0^\infty}_{=0} - \int_0^\infty \left(-\frac{1}{R}e^{-Rt}\right)R^2 dt$$

(use integration by parts, with u=R^{2}t, dv=e^{-Rt}dt)

$$= R\int_0^\infty e^{-Rt}dt = -e^{-Rt}\Big|_0^\infty = 1$$

(The probability is 1 that the interval between firings is between 0 and ∞)

$$\int_0^\infty tp(t)dt = \int_0^\infty R^2 t^2 e^{-Rt}dt$$

$$= \underbrace{R^2 t^2\left(-\frac{1}{R}e^{-Rt}\right)\Bigg|_0^\infty}_{=0} - \int_0^\infty \left(-\frac{1}{R}e^{-Rt}\right)2R^2 tdt$$

(use integration by parts, with u=R^{2}t^2, dv=e^{-Rt}dt)

$$= \frac{2}{R}\int_0^\infty R^2 te^{-Rt}dt = \frac{2}{R}$$

(This is the average interval between firings)

(d) (**2'**) Derivation:

The time interval T between an event and the second following event is the sum of the time intervals between the event and the first following event (say *t*) and the time interval between the first following event and the second following event (say *t'*), i.e. $T = t + t'$. The probability distributions for *t* and *t'* are given:

$$p_1(t) = \mathrm{R}e^{-Rt}, \qquad t > 0$$

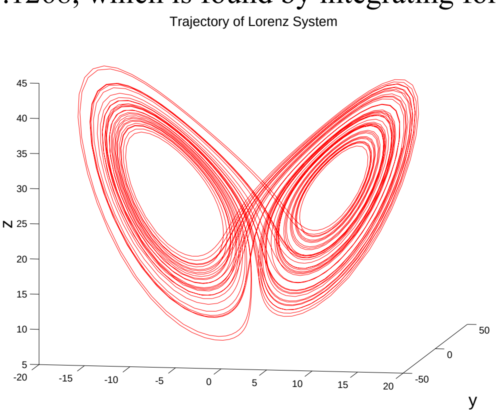
For a Poisson process every event is independent. Thus for a Poisson process, *t* and *t'* are independent random variables and therefore T is the sum of two independent random variables. Thus we can find the probability distribution $p(T)$ for the time interval between an event and the second following event in a Poisson process by using the convolution integral. Since *t* must lie between the event and *T*, the limits of integration for *t* are from 0 to *T*.

$$\begin{aligned} p(T) &= \int_0^T dt\, \mathrm{R}e^{-R(T-t)}\, \mathrm{R}e^{-Rt} \\ &= R^2 e^{-RT} \int_0^T dt \\ &= R^2 T e^{-RT} \end{aligned}$$

4. (**8'**) This project is to examine the Lorenz equations.

(a) (**1'**) Δ = 0.05 is too large. Choose a small enough Δ for your simulation (not larger than 0.01 is OK). Here we use Δ = 0.005.

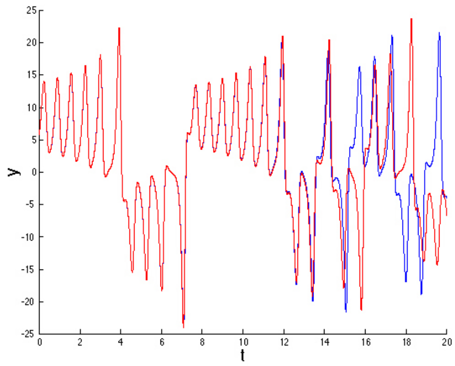
(b) (**2'**) Below is seen a trajectory on the attractor. The initial condition is $x = 4.9022$, $y = 5.7704$, $z = 21.1268$, which is found by integrating for $t = 30$:



(c) (**2'**) Below we see two traces of *y* vs *t* for a trajectory on the attractor. The initial conditions are:

$x = 4.9022,\ y = 5.7704,\ z = 21.1268$ (blue)
$x = 4.9021,\ y = 5.7704,\ z = 21.1268$ (red)

We see that the two values diverge at around t = 13, even though the initial conditions differ by only 0.0001 in one coordinate. This is an example of "sensitive dependence on initial conditions", popularly known as the "butterfly effect." The closer the trajectories start out, the longer it takes for them to diverge.



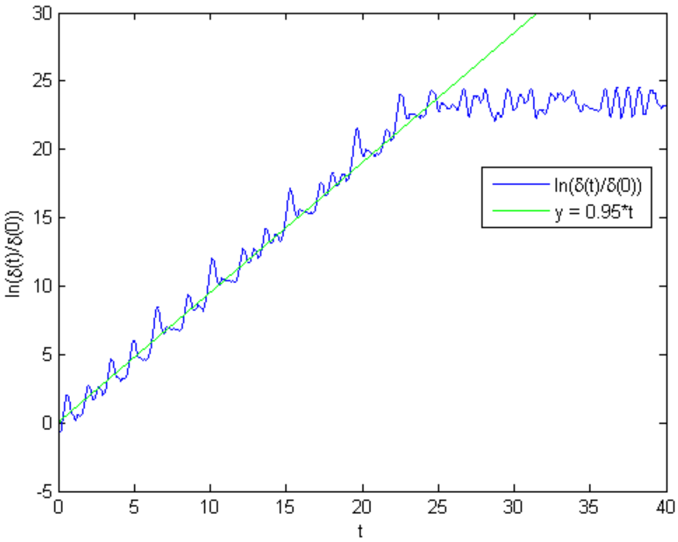
(d) (**3**') The Lyapunov exponent for a strange attractor measures the average rate at which two nearby trajectories exponentially diverge. It is found according to the equation,

$$|\vec{\delta}(t)| \approx |\vec{\delta}(0)| e^{\lambda t}$$

where $\delta(t)$ is the Euclidean distance between the two trajectories at time t. Taking the natural logarithm of both sides of the above equation yields

$$ln\left(\frac{|\vec{\delta}(t)|}{|\vec{\delta}(0)|}\right) \approx \lambda t$$

Thus by plotting the logarithm of the separation between the two trajectories, we expect to get a line with slope λ. Eventually the separation saturates when it reaches the size of the attractor, i.e. since the trajectories are both confined to the attractor, their separation cannot increase beyond the size of the attractor.



From the plot, we estimate the Lyaponov exponent to be $\lambda = 0.95$.

```
clear all;
a = 10;
b = 28;
c = (8/3);
maxtime = 30;
dt = 0.005;
nsteps = maxtime/dt;
t(1) = 0;
epsilon = 0.0001;
%x = 1;
%y = 1;
%z = 1;
%for i = 1:nsteps     % Get onto the attractor
%    x = (a*(y - x))*dt + x;
%    y = (x*(b - z)-y)*dt + y;
%    z = (x*y - c*z)*dt + z;
%end
x1(1) = 4.9022;
y1(1) = 5.7704;
z1(1) = 21.1268;
x2(1)= x1(1) % + epsilon;
y2(1)= y1(1) + epsilon
z2(1)= z1(1) % + epsilon;
d(1) = sqrt((x1(1)-x2(1))^2+(y1(1)-y2(1))^2+(z1(1)-z2(1))^2);
for i = 1:nsteps
    t(i+1) = dt*i;
    x1(i+1) =(a*(y1(i) - x1(i)))*dt + x1(i);
    y1(i+1) = (x1(i)*(b - z1(i))-y1(i))*dt + y1(i);
    z1(i+1) = (x1(i)*y1(i) - c*z1(i))*dt + z1(i);
    x2(i+1) = (a*(y2(i) - x2(i)))*dt + x2(i);
    y2(i+1) = (x2(i)*(b - z2(i)) - y2(i))*dt + y2(i);
    z2(i+1) = (x2(i)*y2(i) - c*z2(i))*dt + z2(i);
    d(i+1) = sqrt((x1(i)-x2(i))^2+(y1(i)-y2(i))^2+(z1(i)-z2(i))^2);
end
plot (t,log(d/d(1)))
```