

MOL 410/510: Biological Dynamics

Fall 2008

Problem Set #6

Solutions

1. Mutual inhibition (12')

We determine the number and type of fixed points by plotting the nullclines and vector field for the system, as shown in Figure 1 (2'). For $\beta = 0.5$, there is a single stable fixed point at $(1/2, 1/2)$ (2'). For $\beta > 1$, there are three fixed points: an unstable fixed point at $(1/2, 1/2)$ and two stable fixed points that approach $(0, 1)$ and $(1, 0)$ (4'). These two stable steady states represents one neuron fully active and the other neuron completely turned off.

To prove this solidly, the fixed points are solutions to equations:

$$x_1 = f(x_2) = \frac{1}{1 + e^{4\beta(x_2 - 0.5)}} \quad (1)$$

$$x_2 = f(x_1) = \frac{1}{1 + e^{4\beta(x_1 - 0.5)}}. \quad (2)$$

Thus we need to solve:

$$\frac{1}{1 + e^{4\beta\left(\frac{1}{1 + e^{4\beta(x_1 - 0.5)}} - 0.5\right)}} - x_1 = 0. \quad (3)$$

There is no analytical solution to Equation (3). Alternatively, we can solve it numerically in Matlab. One method is to use **fsolve** or **fzero**. To call this function, we have to start from some initial value as a guess for the root. From Figure 1, we can easily tell that the roots are around 0, 0.5, and 1.

```
b = 2;  
z1 = fzero(@(x) 1/(1+exp(4*b*(1/(1+exp(4*b*(x-0.5)))-0.5)))-x, 0);  
z2 = fzero(@(x) 1/(1+exp(4*b*(1/(1+exp(4*b*(x-0.5)))-0.5)))-x, 0.5);  
z3 = fzero(@(x) 1/(1+exp(4*b*(1/(1+exp(4*b*(x-0.5)))-0.5)))-x, 1);
```

The Jacobian is:

$$J = \begin{bmatrix} -1 & \frac{-4\beta}{[1+e^{4\beta(x_2-0.5)}][1+e^{-4\beta(x_2-0.5)}]} \\ \frac{-4\beta}{[1+e^{4\beta(x_1-0.5)}][1+e^{-4\beta(x_1-0.5)}]} & -1 \end{bmatrix} \quad (4)$$

It can be proved that $1 - f(x) = f(1 - x)$ (try to prove this by yourself!), so that we can easily get $x_1 + x_2 = 1$ and

$$\frac{-4\beta}{[1 + e^{4\beta(x_2 - 0.5)}][1 + e^{-4\beta(x_2 - 0.5)}]} = \frac{-4\beta}{[1 + e^{4\beta(x_1 - 0.5)}][1 + e^{-4\beta(x_1 - 0.5)}]}, \quad (5)$$

which we rewrite as B .

Accordingly, the eigenvalues are

$$\lambda_{1,2} = -1 \pm |B|. \quad (6)$$

For fixed point $(0.5, 0.5)$, $B = -\beta$. When $|\beta| < 1$, the two eigenvalues are both negative and this fixed point is stable. When $|\beta| > 1$, one eigenvalue is positive and the fixed point is a saddle.

For the other two fixed points (if they exist), we can prove that $|B| < 1$ regardless of β and x values (try to prove this by yourself!). So the other two fixed points are always stable as long as they exist.

The bifurcation diagram for this system as a function of β is shown in Figure 2 (**1'**). From the diagram, it is clear that this is a pitchfork bifurcation.

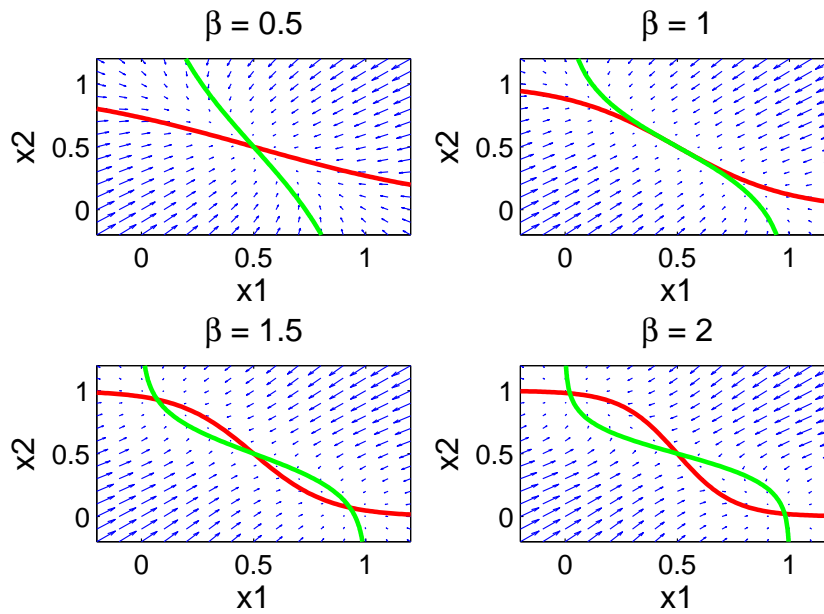


Figure 1: Vector field plots for two mutually inhibiting neurons for different values of β . The x_1 and x_2 nullclines are in red and green, respectively.

When we give these two neurons different stimuli k_1 and k_2 , we find that (for $\beta > 1$) we go from having two stable states to having just one stable state (**1'**), as shown in Figure 3. This single stable steady state corresponds to the neuron receiving the larger stimulus being fully active and the neuron receiving the smaller stimulus being completely inactive (**1'**). Thus, whichever neuron receives the larger stimulus always “wins” by becoming maximally active, while fully repressing the other neuron. In this way a group of mutually inhibiting neurons behave as a “winner-take-all” network (**1'**).

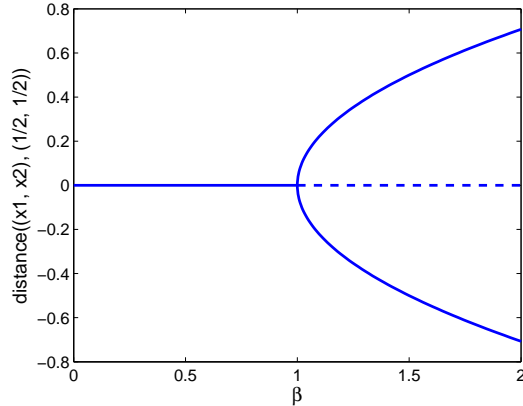


Figure 2: Bifurcation diagram for two mutually inhibiting neurons as a function of β .

2. Fitzhugh-Nagumo (8')

- The fixed points are where $dv/dt = 0$ and $dw/dt = 0$. The second equation gives us $v = \gamma w$. Substituting into the first, we have (1')

$$v(v - a)(v - 1) + v/\gamma - I = 0. \quad (7)$$

An example of a Matlab code that finds the fixed point and plots the real part of the eigenvalues as a function of the input is shown below. We used the values for the parameters as given in the text. First we define the functions f and g . Then we find the fixed point by solving Equation (7) (1'). The a cubic equations, we can find three roots. But only the real one makes sense (membrane potential v and recovery variable w cannot be complex numbers). Thus we picked the first root and discard the other two. Next we find the Jacobian matrix at the fixed point, and then find the eigenvalues of the matrix (2'). The graph of the real part of the eigenvalues shown in Figure 4 indicates that the real part becomes positive at $I = 0.03507$ (1'), which is the value at which the fixed point becomes unstable.

```
syms I v w
a = 0.139; eps = 0.008; gamma = 2.54;
f = I - v * (v-a) * (v-1) - w;
g = eps * (v - gamma * w);
h = v*(v-a)*(v-1) + v/gamma - I;
fp = solve( h );
J = [diff(f,v), diff(f,w); diff(g,v), diff(g,w)];
Jac = subs(J,v,fp(1));
E = eig(Jac);
figure,
ezplot(real(E(1)), [0, 0.1])
```

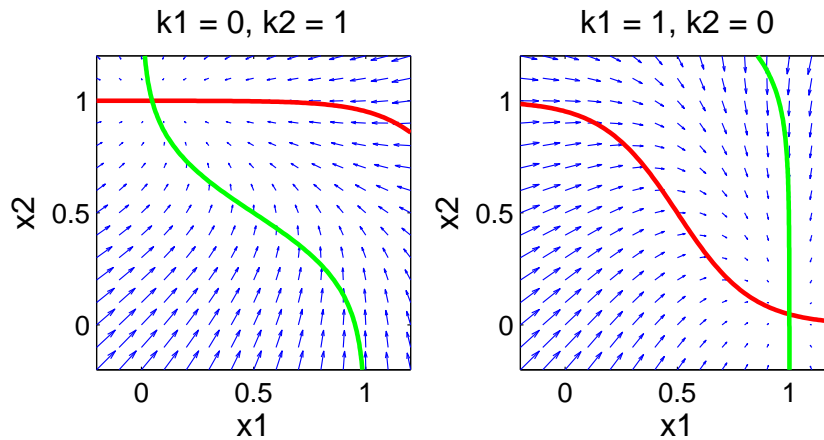


Figure 3: Vector field plots for two mutually inhibiting neurons for different stimulus strengths k_1 and k_2 . On the left neuron 2 receives the larger stimulus and on the right, neuron 1 receives the larger stimulus. The x_1 and x_2 nullclines are in red and green, respectively.

```
hold on
ezplot(real(E(2)), [0, 0.1])
```

- Graphs of the trajectory and membrane potential v as a function of time t are shown in Figure 5 (2'). In this case, we turned I on at $t = 10$, and off at $t = 20$. If I is kept on all the time, there will be a limit cycle (1'). Instead of single potential pulse, v is going to oscillate.

A Matlab code for plotting the graphs is as following:

```
function trajectory()
% Show the trajectory for current I
I = 0.1;
a = 0.139; eps = 0.008; g = 2.54;
time = 200; % time to run to
t1 = 10; % time to turn on
t2 = 20; % time to turn off

x0 = [0 0];
[T x] = ode45(@(t,x) myfun(t,x,I,a,eps,g,t1,t2), [0 time], x0);

figure, plot(x(:,1),x(:,2));
figure, plot(T, x(:,1));

function y = myfun(t, x, I, a, eps, g, t1, t2)
```

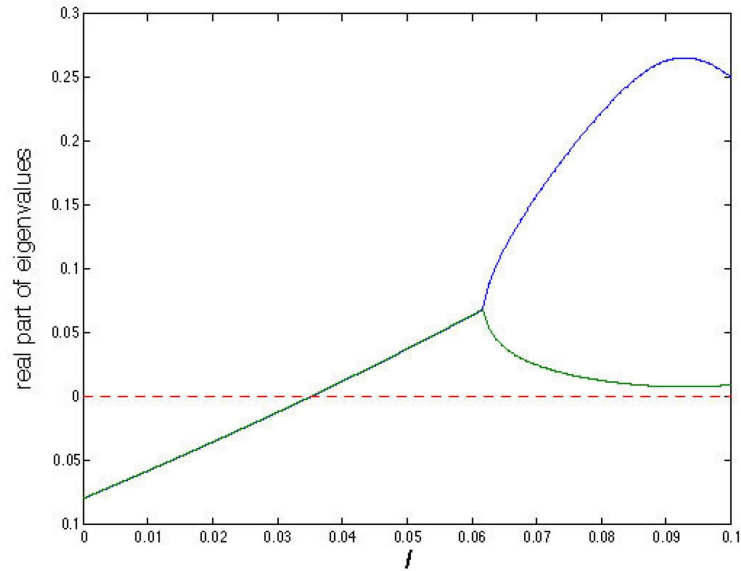


Figure 4: The real part of the eigenvalues of the linearized dynamics at the fixed point versus the current I injected into the axon.

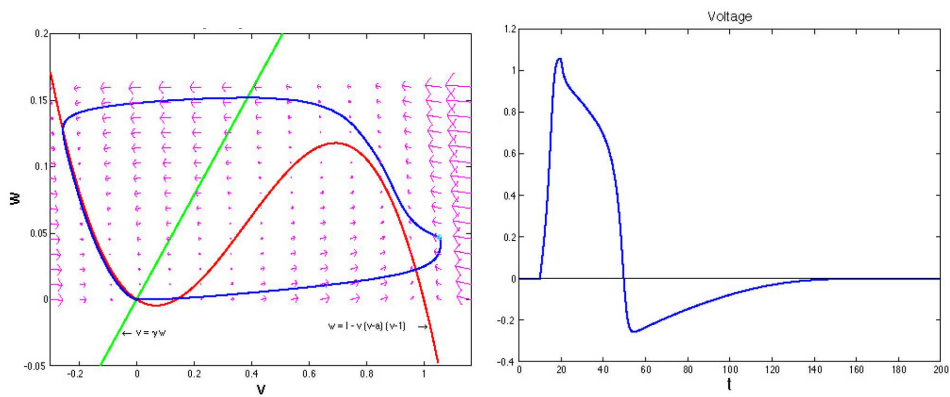


Figure 5: Left: The trajectory (blue) on top of the vector field (pink). Red and green are v -nullcline and w -nullcline respectively. Right: Membrane potential v as a function of time t . Injected current I is kept on only from $t = 10$ to $t = 20$.

```
        y = [ I*(t>t1)*(t<t2) - x(1)*(x(1)-a)*(x(1)-1) - x(2);  
              eps*(x(1) - g*x(2))];  
    end  
end
```