Introduction to Biological Dynamics
Homework 0 – September 15, 2011

This is a non-graded homework that will help you refresh your background on Taylor series and complex numbers.

1. **Taylor expansions** (If you read this question and realize you remember this well from college, you can skip the question. There's a long intro, to teach you about Taylor expansions. The actual questions are after the • signs.)

   Any continuous differentiable function can be well approximated by it's *Taylor expansion*, meaning an expansion in powers of x around some point x_0 at which you know the derivatives. Let's go through it.

   Suppose you're thinking of the function $f(x)$. Then at points close to x_0, the first shot at an approximation would be that $f(x)$ is $\approx$ to its value at x_0:

$$\hat{f}(x) \approx f(x_0) \qquad \text{(we'll use } \hat{f} \text{ to mean "approximation of f")}$$

   As you get further away from x_0, that approximation is clearly going to get worse and worse. One thing you could do is fit a straight line that goes through $f(x_0)$, and that is tangent to $f(x)$ at x_0– in other words, a line has the same derivative as $f(x)$ at $x = x_0$. That would provide an approximation that is a little better than plain old $f(x_0)$ for values of x a little bit further away from x_0. If you do that, you're saying that your approximation is:

$$\hat{f}(x) \approx f(x_0) + (x - x_0) \cdot \frac{df}{dx}|_{x_0} \tag{1}$$

   where $\frac{df}{dx}|_{x_0}$ means "some number, i.e., a constant, that is equal to the first derivative of f with respect to x, evaluated at $x = x_0$." The approximation in equation (1) is called a *first-order approximation*.

   • Differentiate equation (1) with respect to x, and evaluate it at $x = x_0$ to show that $\hat{f}$ has the same derivative as f at $x = x_0$. Show that $\hat{f}$ also has the same value as f at $x = x_0$.

   That got you somewhere, but you could go further. Suppose you wanted to approximate f with a function that, as above, (a) matches the value of f at $x = x_0$; (b) matches the first derivative of f at $x = x_0$; but now, in addition, (c) matches the second derivative of f at $x = x_0$. In other words, we now also want to match the curvature of the function f at the point where $x = x_0$. Then you could construct the following polynomial that has up to quadratic powers of x:

$$\hat{f}(x) \approx f(x_0) + (x - x_0) \cdot \frac{df}{dx}|_{x_0} + \frac{1}{2}(x - x_0)^2 \cdot \frac{d^2 f}{dx^2}|_{x_0} \tag{2}$$

• Show that $\hat{f}$ of equation (2) has the same value as f at $x = x_0$, differentiate it to show that it has the same derivative as f at $x = x_0$, and differentiate it again to show that it has the same second derivative as f at $x = x_0$.

We hope you're beginning to see the pattern here! Let's do one more round, and then we'll go to the general equation. Take the cubic polynomial

$$\hat{f}(x) \approx f(x_0) + (x - x_0) \cdot \frac{df}{dx}|_{x_0} + \frac{1}{2}(x - x_0)^2 \cdot \frac{d^2 f}{dx^2}|_{x_0} + \frac{1}{3!}(x - x_0)^3 \cdot \frac{d^3 f}{dx^3}|_{x_0} \tag{3}$$

• and show that $\hat{f}$ from equation (3) matches the value of f as well as its first, second, and third-order derivatives at $x = x_0$.

Ok, we're ready to go whole hog! Take

$$\hat{f}(x) \approx \sum_{n=0}^{\infty} \frac{1}{n!}(x - x_0)^n \cdot \frac{d^n f}{dx^n}|_{x_0} \tag{4}$$

and show that $\hat{f}$ matches f in value and *every* derivative (i.e., first, second, etc.). Equation (4) is the Taylor expansion of f around $x = x_0$.

Finally, let's do a specific example in Matlab. Take the function

$$g(x) = \frac{1}{1 + e^{-x^3}} \tag{5}$$

Plot it from $x = -2$ to $x = 2$. Remembering your calculus, you can manually compute the first, second, third, etc., derivatives of this function. But, to keep you focused on Taylor expansions rather than the minutae of keeping track of coefficients as you differentiate, we've done it for you. Download and unzip `gfunc.zip` from the wiki. That contains a file, `gfunc.m`, that computes g and its first through fourth derivatives.

• Write a script that uses `gfunc.m` to plot g and its first, second, third, and fourth-order Taylor approximations around some point x_0. For your first choice of x_0, use $x_0 = -0.35$. Then choose other values, whatever looks interesting to you (for example, $x_0 = +0.5$).

Comment on what you observe. Do the approximations get better as you use more terms? Over what range would the approximation be a good one if you used an infinite number of terms? If you used an infinite number of terms, would the choice of x_0 matter?

2. **De Moivre's Theorem and The Best Equation EVER**

- Remembering that

$$\frac{de^x}{dx} = e^x$$

show that the Taylor expansion of e^x around $x = 0$ is

$$e^x = 1 + x + \frac{1}{2}x^2 + ... + \frac{1}{n!}x^n + ... \tag{6}$$

- Remembering that

$$\frac{d\sin\theta}{d\theta} = \cos\theta \quad \text{and} \quad \frac{d\cos\theta}{d\theta} = -\sin\theta$$

Show that the Taylor expansion of $\cos\theta$ and $\sin\theta$ around $\theta = 0$ are

$$\begin{aligned} \cos\theta &= 1 - \frac{1}{2}\theta^2 + \frac{1}{4!}\theta^4 - \frac{1}{6!}\theta^6 + ... \text{ etc} \\ \sin\theta &= \theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \frac{1}{7!}\theta^7 + ... \text{ etc} \end{aligned} \tag{7}$$

- Remembering that

$$i^2 = -1$$

show that the Taylor expansion of $e^{i\theta}$ equals the sum of the Taylor expansions of $\cos\theta$ plus $i\sin\theta$. Thus

$$e^{i\theta} = \cos\theta + i\sin\theta$$

This hugely important equation is known as *De Moivre's* theorem. It beautifully and astonishingly uses an imaginary number to link two completely separate irrational numbers. The number e and the number π are both irrational. Each require an infinite number of decimal points to write them down. They are defined completely separately and independently of each other. Yet De Moivre's theorem proves that they are perfectly linked to each other. Letting $\theta = \pi$, we gobsmackingly get:

$$e^{i\pi} = -1 \tag{8}$$