MOL 410/510: Introduction to Biological Dynamics Fall 2012
Problem Set #3 SOLUTIONS

1. **Brobdingnag**.

   (a) If $p(t)$ is the population of Brobdingnag, then the number of births is given by $bp(t)$ and the number of deaths is $dp(t)$. Therefore, assuming that only births and deaths contribute to the change in population, then the differential equation is $\dot{p}(t) = (b - d)p(t)$.

   (b) When $d > b$, more people die than are born each year, therefore the population will eventually become extinct and $p = 0$ is stable. If on the other hand $d < b$, then there will be more births than deaths and there will be a population explosion; i.e. $p = 0$ is unstable.

   (c) The solution to the differential equation yields $p(t) = p_0 e^{(b-d)t}$, where p_0 is the initial population. Since we want to calculate the time for the population to double from today, we'll say the current population is p_0, i.e. right now $t = 0$. Now, let's find the time t_2 at which the population will double, i.e. $p(t_2)/p_0 = 2 = p_0 e^{(b-d)t_2}/p_0 = e^{(b-d)t_2}$. Therefore, $t_2 = \ln 2/(b - d) = \ln 2/(0.1) \approx 6.9$ years.

   It turns out that Brobdingnagians are choosing to have very few children these days, with the result that although d remains at $d = 0.1$, the value of b has fallen to $b = 0.05$. In response, the Brodingnaginian king has started encouraging immigration, which comes in at a constant rate of m people per year.

   (d) The differential equation is now $\dot{p}(t) = (b - d)p(t) + m$. To answer the second part of the equation, let's set $\dot{p} = 0$. The population will then stabilize at the value $p_\infty = m/(d - b)$; therefore, $m = p_\infty(d - b) = 20 \times 10^6(0.05) = 1$ million people per year.

   (e) To solve the differential equation, you can separate dx and dt and integrate:

   $$\int \frac{dp}{(b-d)p + m} = \int dt$$

   If you have trouble performing the integral on the left side of the equal sign, use a substitution, e.g. $z = (b - d)p + m$ and $dz = (b - d)dp$. The final solution is

   $$p(t) = \frac{e^{(b-d)t}(p_0(b - d) + m) - m}{b - d}$$

   See attachment with code for the remainder of the solution.

2. **Oscillations**

   The equation has the simple analytical solution $x = x_0 e^{\lambda t}$, this is true regardless of the value of λ. However, the nature of the dynamics will be completely different for different kinds of λ. See the Matlab code for the solution.

3. **Diagonal multi-dimensional dynamics**

   (a) $\mathbf{x} = 0$ is stable when all λ_i are real and less than 0. If some are less than 0 while others are greater than 0, then the fixed point is a saddle point; dimensions where the eigenvalues are greater than 0, the dynamics are unstable.

   (b) Plugging in $V\mathbf{y}$ for $\mathbf{x}$, we get $V\dot{\mathbf{y}} = AV\mathbf{y}$ or $\dot{\mathbf{y}} = V^{-1}AV\mathbf{y} = D\mathbf{y}$. Therefore the differential equation is $\dot{\mathbf{y}} = D\mathbf{y}$. In order for $\mathbf{y} = 0$ to be a stable point, all elements of D must be less than 0. The same is true for $\mathbf{x} = 0$ to be a stable point.

4. **Multidimensional linear dynamics.**

$$\begin{aligned} \dot{x} &= -1.25x + 1.3y, \\ \dot{y} &= 1.3x + 0.25y \end{aligned}$$

   (a) The matrix equation is

$$\dot{\mathbf{u}} = \begin{pmatrix} -1.25 & 1.3 \\ 1.3 & 0.25 \end{pmatrix} \mathbf{u}$$

   (b) See attached code.

   (c) See attached code.

   (d) See attached code.

   (e) See attached code.