MOL 410/510: Introduction to Biological Dynamics Fall 2012
Problem Set #5, Nonlinear dynamics and limit cycles (Extra Credit) (due 10/26/2012)
1 Questions

Clearly explain your answers, show all your work, and include any figures or code where requested. Don't forget that you can use Matlab's help function to get information on any of the functions introduced in this homework.

1. **Chemical Reaction**.

   The most straightforward way to prove the existence of a limit cycle is through the application of the Poincare-Bendixson theorem. This theorem asserts that, for a closed and bounded region of the plane, if no trajectories can leave the region as they settle down (i.e., they stay there forever after) and the region contains no equilibria, that region must contain (or consist exclusively of) a limit cycle.

   Thus, we wish to demonstrate that there is a bounded region of the plane from which no trajectory can leave given the correct parameters. Thus we need to show that the fixed point is unstable and a region around the fixed point is attracting. If this is the case, all non-equilibrium trajectories that enter the region stay in it forever, and the theorem is satisfied.

   We start by finding the nullclines. These are given by

$$\begin{aligned} y &= \frac{(a-x)(1+x^2)}{4x} \\ y &= 1+x^2 \end{aligned}$$

   The first equation is the x-nullcline the second equation is the y-nullcline. The fixed point is therefore given by $x^* = a/5$, $y^* = 1 + a^2/25$. In order for a limit cycle to exist, this fixed point should be unstable. Let's find the conditions for that to be true. As usual, we want to calculate the Jacobian. The derivatives in this case are somewhat complicated, so let's take it step by step.

$$\begin{aligned} \frac{d\dot{x}}{dx} &= \tfrac{d}{dx}\left(a - x - \tfrac{4xy}{1+x^2}\right) &&= -1 - \frac{4y(1+x^2) - 8yx^2}{(1+x^2)^2} \\ \frac{d\dot{x}}{dy} &= \tfrac{d}{dx}\left(a - x - \tfrac{4xy}{1+x^2}\right) &&= -\frac{4x}{1+x^2} \\ \frac{d\dot{y}}{dx} &= \tfrac{d}{dx}\left(bx - \tfrac{bxy}{1+x^2}\right) &&= b - \frac{by(1+x^2) - 2bx^2y}{(1+x^2)^2} \\ \frac{d\dot{y}}{dy} &= \tfrac{d}{dy}\left(bx - \tfrac{bxy}{1+x^2}\right) &&= -\frac{bx}{1+x^2} \end{aligned}$$

   At this point, we could just plug in the values for x^* and y^* and work it all out. However, it's simpler to make use of the fact that $y^* = 1 + x^{*2}$. Since, we are calculating the Jacobian about the fixed point, everywhere we see $(1+x^2)$ in the derivatives, we can replace that by y. This reduces the derivatives to

$$\begin{aligned}
\left.\frac{d\dot{x}}{dx}\right|_{x^*,y^*} &= \frac{-5+3x^{*2}}{y*} \\
\left.\frac{d\dot{x}}{dy}\right|_{x^*,y^*} &= -\frac{4x^*}{y^*} \\
\left.\frac{d\dot{y}}{dx}\right|_{x^*,y^*} &= \frac{2bx^{*2}}{y^*} \\
\left.\frac{d\dot{y}}{dy}\right|_{x^*,y^*} &= -\frac{bx^*}{y^*}
\end{aligned}$$

Therefore the trace and determinant are given by

$$\begin{aligned}
\tau = \quad \tfrac{3x^{*2}-bx^*-5}{y^*} \quad &= \frac{3a^2/25-ba/5-5}{1+a^2/25} \\
\Delta = \quad \tfrac{5bx^{*3}+5bx^*}{y^{*2}} \quad &= \frac{5b(a^3/125+a/5)}{(1+a^2/25)^2}
\end{aligned}$$

In order for the fixed point to be unstable, we require that $\tau > 0$ and $\Delta > 0$. This will happen when $3a^2 - 5ba - 125 > 0$, $a^2 > -25$, and $b > 0$. If $b < 0$, the fixed point is a saddle point, and there can be no stable limit cycle.

The only thing left is to find a trapping region. We're going to find a rectangular region where all of the flow must return inside the box. Let's first look at the vertical line defined by $x = 0$. Along that line, $\dot{y} = 0$ and $\dot{x} = a$. So long as $a > 0$, then the flow will be back into the box and only towards the right. Since the x-coordinate of the fixed point is given by $x^* = a$ and we know that we are in the positive quadrant, $a > 0$.

Now let's examine the line at $y = 0$. There, $\dot{x} = a - x$ and $\dot{y} = bx$. If $b > 0$, then all points along that line flow upward (except (0,0) which flows purely to the right). But we know $b > 0$, otherwise the determinant of the Jacobian is negative and the fixed point is a saddle point, destroying the limit cycle. Looking at the direction of flow in the x-direction, all points up to $x = a$ flow to the right, past that all points flow back to the left.

That leads us to look at the vertical line $x = a$ as another wall to our trapping region. There, $\dot{x} = -4ay/(1+a^2)$ and $\dot{y} = ba(1-y/(1+a^2))$. All flow is to the left along this line, except at $y = 0$ where it flows up only. Also, the flow is always up along this line up to $y = 1 + a^2$, where the flow flips to downward.

Therefore, we will now look at the horizontal line given by $y = 1 + a^2$. There, $\dot{x} = a - x - 4x(1+a^2)/(1+x^2)$ and $\dot{y} = bx(1-(1+a^2)/(1+x^2))$. Since, we are interested in points where $x \leq a$, i.e. inside the bounding box, we can deduce that $\dot{y} \leq 0$ along this line (that is, $(1 + a^2)/(1 + x^2) \geq 1$ when $x \leq a$). Therefore, all arrows point down and we have defined a bounding box.

We now have an unstable fixed point and a trapping region around it. Any trajectory in this region of the plane must stay within it, and the region contains no equilibria. Hence, by the Poincare-Bendixson theorem, the region must contain a limit cycle for the parameter values described. Q.E.D.