PREPARED BY
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Nonlinear Dynamics / Bifurcations II

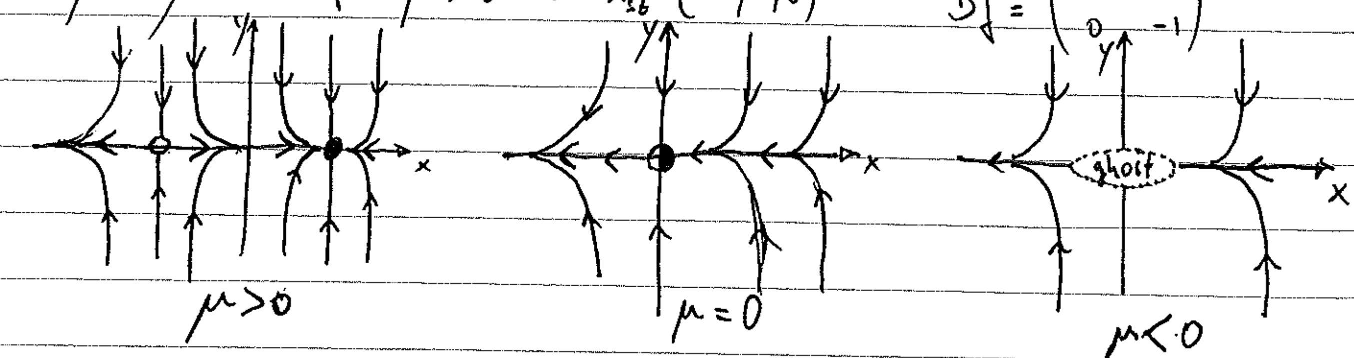
Bifurcations in 2(N)-dimensions:

→ nothing much new happens, action is confined to 1D subspace along which bifurcation occurs; extra-dimension has only attractive or repulsive flow. (1D bifurcation are building blocks for high D)

saddle-node bifurcation in 2D (basic mechanism for creation/destruction of fixed points)

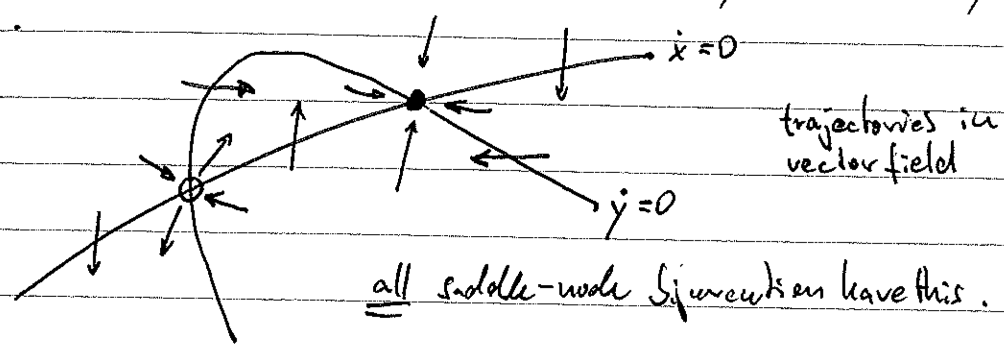
$\dot{x} = \mu - x^2$

$\dot{y} = -y$ for $\mu > 0 \Rightarrow \vec{x}_{st} = (\pm\sqrt{\mu}, 0)$ $\quad Df = \begin{pmatrix} -2x & 0 \\ 0 & -1 \end{pmatrix}$



In x-direction we see bifurcation behavior as before, in y exponentially damped motion.

$\dot{x} = f(x,y)$
$\dot{y} = g(x,y)$

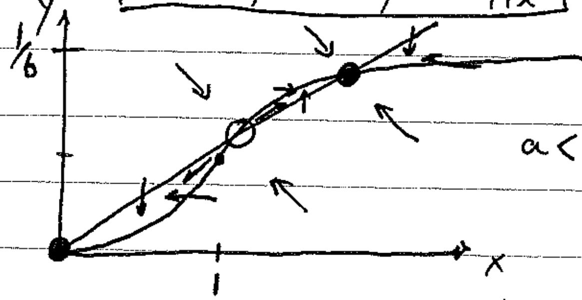


<u>all</u> saddle-node bifurcations have this.

Revisit bi-stable genetic switch:

$$\boxed{\begin{aligned} \text{protein: } \dot{x} &= -ax + y \\ \text{mRNA: } \dot{y} &= -by + \frac{x^2}{1+x^2} \end{aligned}}$$ (dimensionless)

nullclines: $y = ax$
$y = \frac{x^2}{b(1+x^2)}$



$a < a_c \rightsquigarrow$ 3 fixed points

fixed points: $ax = \frac{x^2}{b(1+x^2)} \Rightarrow x_{st} = 0 \quad x_{st} = \frac{1 \pm \sqrt{1-4a^2b^2}}{2ab} \Rightarrow a_c = \frac{1}{2b}$

stability: $Df = \begin{pmatrix} -a & 1 \\ \frac{2x}{(1+x^2)^2} & -b \end{pmatrix}$ $\quad Tr = -(a+b) < 0 \Rightarrow$ sinks or saddles

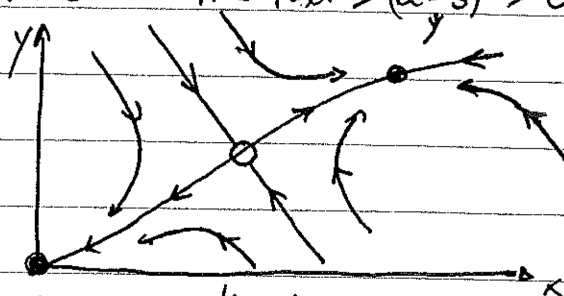
At $(0,0)$, $\det = ab > 0 \Rightarrow$ origin is always stable fixed point

For other two f.pts: $\det = ab - \frac{2x_{st}}{(1+x_{st}^2)^2} = ab\left(\frac{x_{st}^2 - 1}{x_{st}^2+1}\right)$

$\Rightarrow$ "middle" fixed pt. $0 < x_{st} < 1 \longrightarrow$ saddle point $(\det < 0)$; unstable

$\Rightarrow$ $x_{st} > 1$ fixed point $\longrightarrow$ $\det < ab \Rightarrow Tr^2 - 4\det > (a-b)^2 > 0$

$\longrightarrow$ always stable node



The unstable saddle node separates state space into two regions, each with a basin of attraction, the sink.

biological interpretation:

system can act as a biochemical switch if mRNA (b) and protein (a) degrade slowly enough such that $ab < \frac{1}{2}$ (i.e. 3 fixed points).

origin: gene is silent, not enough protein around to turn it on

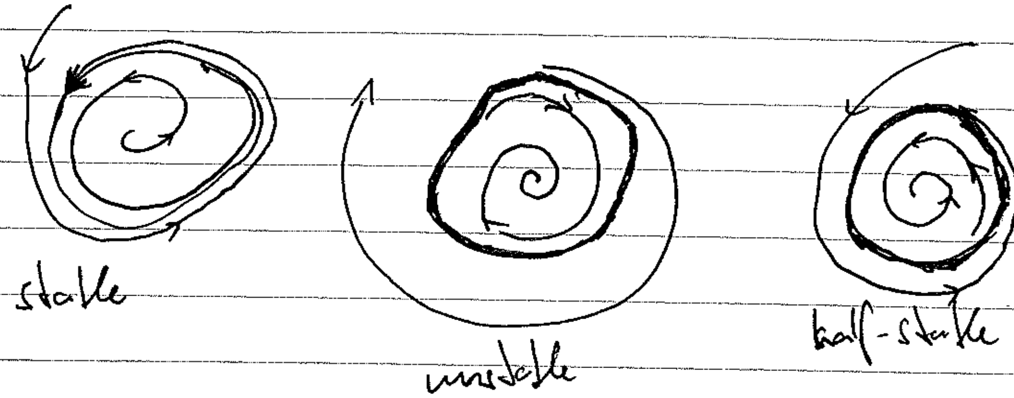
high f.p.: gene is active and sustained by high protein levels

saddle: acts like a threshold switch for ON/OFF states of gene dependent on initial (x_0, y_0) value.

Limit cycles

closed trajectories that are isolated in the vector field

- trajectories never touch cycle
- cycles cannot cross

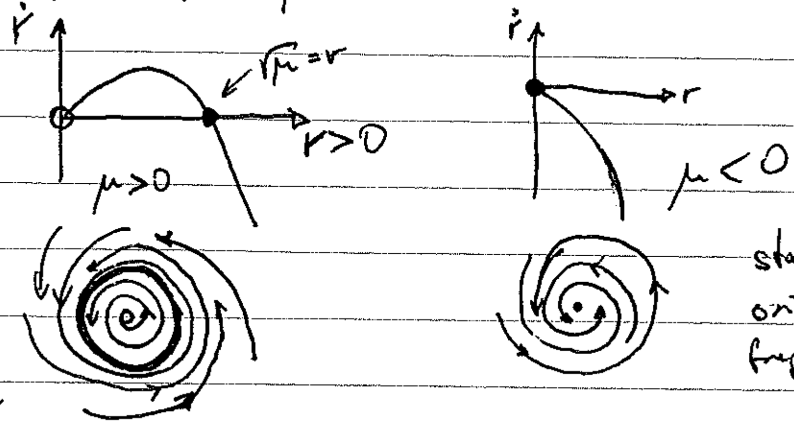


Example: breathing rate and many other periodic examples in Nature.

Existence: no general way to show for arbitrary system...
$\Rightarrow$ use computer (but need to know *how* to look for it)

Example: $\dot{r} = \mu r - r^3$ (1) polar coordinates: $x = r\cos\phi$
$\dot{\theta} = \omega$ (2) $y = r\sin\phi$

$\rightarrow$ 2 fixed points for eq. (1): $r_* = 0$ and $r = \sqrt{\mu}$

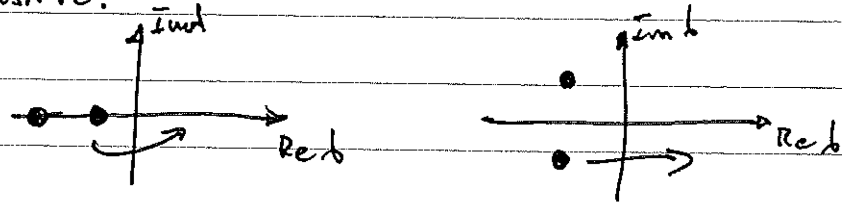


stable limit cycle at $r = \sqrt{\mu}$ and unstable spiral in center

stable spiral orientation and frequency given by ω

Bifurcations in 2D (Hopf Bifurcations)

In 2D a stable fixed point can lose stability by a parameter turning positive or if eigenvalues are complex and either turns their Re part positive.



Consider: $\dot{r} = \mu r - r^3$
$\dot{\theta} = \omega + b r^2$ (b determines the dependence of frequency on amplitude for large amplitude oscillations)

$\Downarrow$

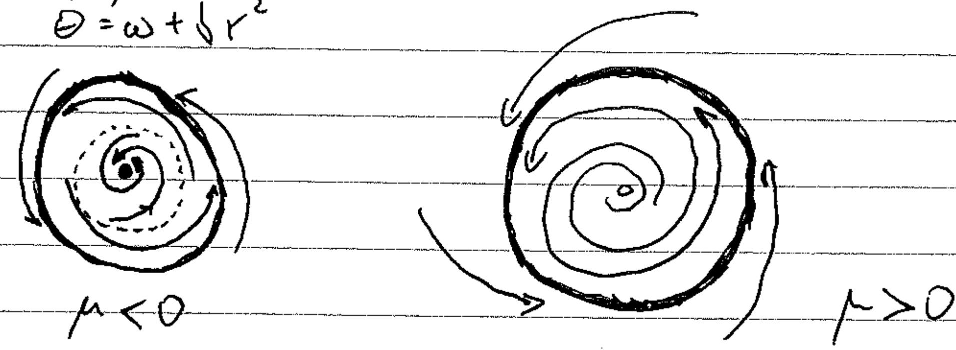
$\dot{x} = \mu x - \omega y$ + cubic terms
$\dot{y} = \omega x + \mu y$ + cubic terms $\Rightarrow$ Jacobian at $(0,0)$: $A = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix}$

$\Rightarrow \lambda = \mu \pm i\omega$ as μ increases from neg to pos, λ cross Im axis

$\Rightarrow$ Supercritical Hopf Bifurcation (with birth of limit cycle)

Like pitchfork bif., Hopf bifurcations have both <u>super-</u> and <u>sub</u>-critical varieties. In the latter trajectories can "jump" to fixed points or limit cycles!

<u>Consider</u>: $\dot{r} = \mu r + r^3 - r^5$
$\dot{\theta} = \omega + b r^2$



At $\mu = 0$ occurs a sub-critical Hopf bifurcation: the unstable limit cycle shrinks to 0 amplitude and engulfs the origin, rendering it unstable. To get a 3D $\dot{x}-x-\mu$ image, rotate the $x-r$ plane of the sub-critical pitchfork ...

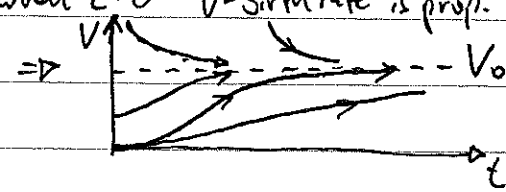
<u>Example</u>: Predator - Prey system (or exploiters E and victims V)

$\dot{E} = a(V - V_c)E$ (food must be larger than threshold V_c)

$\dot{V} = b\,V(V_0 - V)V - c\,EV$ (when $E=0$ V-birth rate is prop. to $V(V_0 - V)$)

make dimensionless
try: $t = T\tau$; $V = v\,x(t)$
$E = e\,y(t)$

$\Rightarrow$



more realistic sigmoidal growth.

$$\dot{x} = T b v^2\left(\frac{V_0}{v} - x\right)x^2 - c e T\, x y$$
$$\dot{y} = T v a\left(x - \frac{V_c}{v}\right)y$$

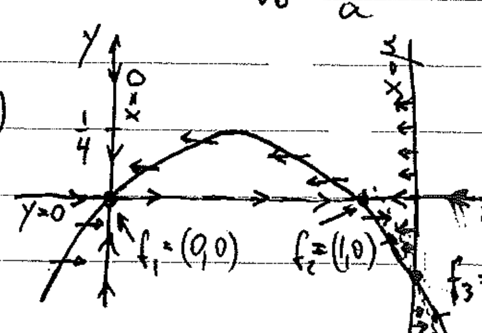
$\xrightarrow[T = \frac{1}{bV_0^2};\ v = V_0;\ e = \frac{1}{cT}]{\text{chose}}$

$$\dot{x} = x^2(1-x) - xy$$
$$\dot{y} = \frac{a}{bV_0}\left(x - \frac{V_c}{V_0}\right)y \equiv (x - \mu)y$$

with $x(t) > 0$; $y(t) > 0$; $\mu > 0$ and $V_0 = \frac{b}{a}$.

Nullclines: $\dot{x} = 0 \Rightarrow \begin{cases} x = 0 \\ y = x(1-x) \end{cases}$

$\dot{y} = 0 \Rightarrow \begin{cases} y = 0 \\ x = \mu \end{cases}$



predators (E or y) go extinct if $\mu > 1$

Stability analysis :

$$J_{f_1} = \begin{pmatrix} 0 & 0 \\ 0 & -\mu \end{pmatrix}$$ half stable node / half saddle point

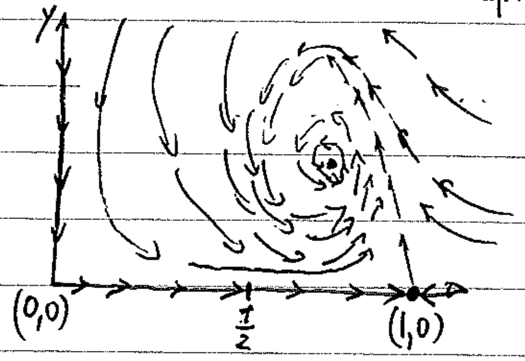
$$J_{f_2} = \begin{pmatrix} -1 & -1 \\ 0 & (1-\mu) \end{pmatrix} \quad \left.\begin{matrix} Tr = -\mu \\ det = \mu - 1 \end{matrix}\right\} \mu > 1 \Rightarrow \begin{cases} X(t) = 1 \\ y(t) = 0 \end{cases} \text{extinction} \quad \begin{matrix} \lambda_1 = -2 \\ \lambda_2 = 2(1-\mu) \end{matrix}$$

$\mu < 1 \Rightarrow f_2$ is a saddle point.

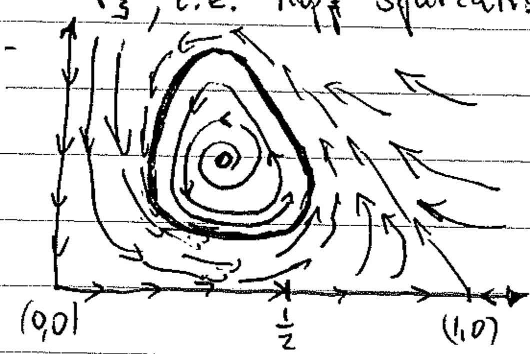
$$J_{f_3} = \begin{pmatrix} \mu(1-2\mu) & -\mu \\ \mu(1-\mu) & 0 \end{pmatrix} \quad \begin{matrix} Tr = \mu(1-2\mu) \\ det = \mu^2(1-\mu) \end{matrix} \Rightarrow \lambda_{1/2} = +\mu(1-2\mu) \pm 2\mu \underbrace{\sqrt{\mu^2-1}}_{<0 \,\Rightarrow\, Im}$$

$\Rightarrow$ there exists a critical $\mu_c = \frac{1}{2}$ where $Re\,\lambda$ switches sign.

$\Rightarrow$ stable or unstable spiral at f_3, i.e. Hopf bifurcation



$\mu > \mu_c = \frac{1}{2}$ stable spiral



$\mu < \mu_c$ unstable f.p. + stable limit cycle

$\Rightarrow$ crucial parameter for system is $\mu = \frac{V_c}{V_0} = \frac{\text{critical pop. for predators to exist}}{\text{steady state prey pop. w.o. pred.}}$

1) $\mu > 1 \Rightarrow$ extinction of predator, no dynamics

2) $\frac{1}{2} < \mu < 1 \Rightarrow$ stable steady state : $V(t) = V_c$, $\left.\begin{matrix} y(t) = \mu - \mu^2 \\ \frac{E(t)}{e} = \frac{V_c}{V_0} - \frac{V_c^2}{V_0^2} \end{matrix}\right\} E(t) = \frac{b}{c} V_c (V_0 - V_c)$

no oscillations for $t \to \infty$

3) $0 < \mu < \frac{1}{2} \Rightarrow V_0 > 2V_c \Rightarrow$ periodic population oscillations

imagine V_c fixed, initially $V_0 < 2V_c$, and parameters a, b, c change w/ time.
V_0 increases and when $V_0 = 2V_c \to$ Hopf ; oscillations occur with frequency $Im\,\lambda = 2\mu\sqrt{1-\mu^2} = \omega$