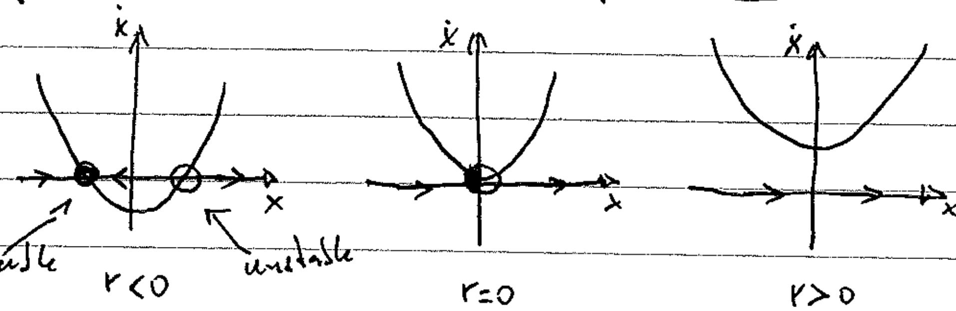
PREPARED BY | DATE 10/11

Nonlinear dynamics / Bifurcations I

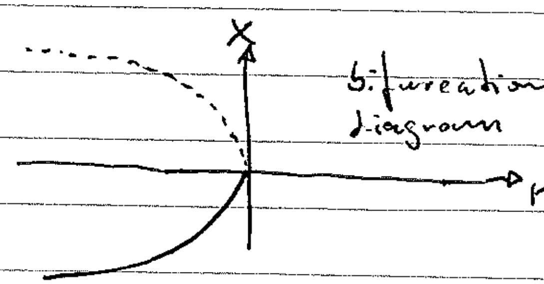
Bifurcation: qualitative change of dynamics upon parameter change

most fundamental bifurcation: **Saddle-node Bifurcation**

$$\boxed{\dot{x} = r + x^2}$$



⇒ bifurcation occurs at $r = 0$
where two fixed points collide

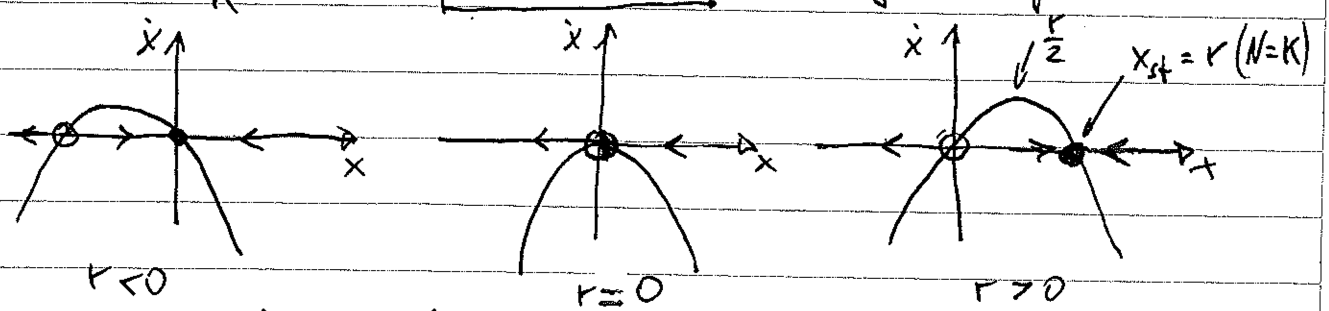


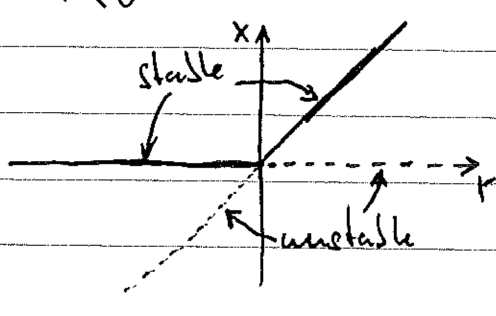
Transcritical Bifurcation:

Consider growth of a population $\dot{N} = rN$ with growth rate r
→ predicts exponential growth $N(t) = N_0 e^{rt}$
→ unrealistic; for populations larger than "capacity" K death is higher than birth: $\dot{N} = rN\left(1 - \frac{N}{K}\right)$ (per capita growth rate $\frac{\dot{N}}{N}$ decreases linearly)

change of variables to make this equation dimensionless:

$$x \equiv \frac{rN}{K} \Rightarrow \boxed{\dot{x} = rx - x^2} \quad \text{logistic equation}$$

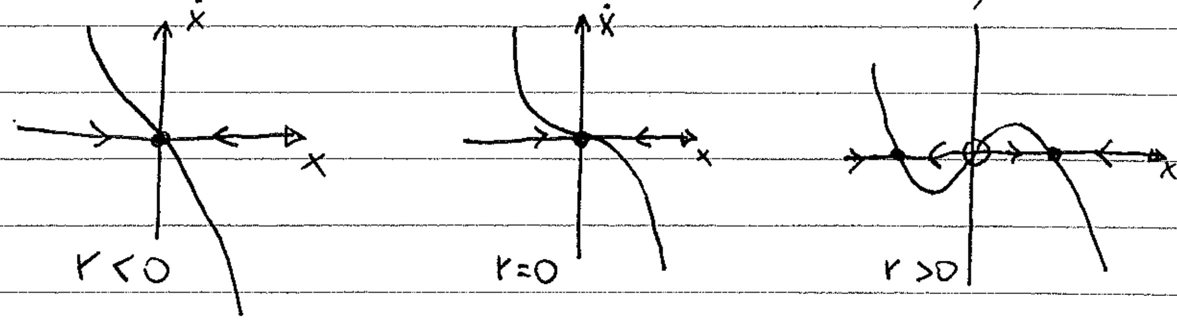




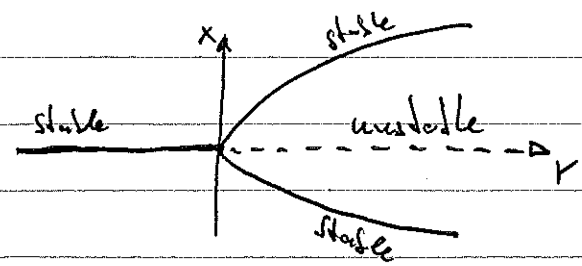
In the transcritical bifurcation, the two fixed points don't disappear (like in saddle-node case); instead they just switch their stability

Pitchfork bifurcation

A) $\dot{x} = rx - x^3$ (supercritical pitchfork bifurcation)

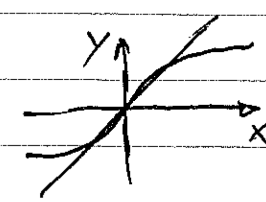


consider $x_{st} = 0$: how stable is it? ⇒ magnitude of $\frac{1}{|f'(x_{st})|}$ indicates characteristic time scale that determines the time required for $x(t)$ to vary significantly around x_{st}



"pitch fork"

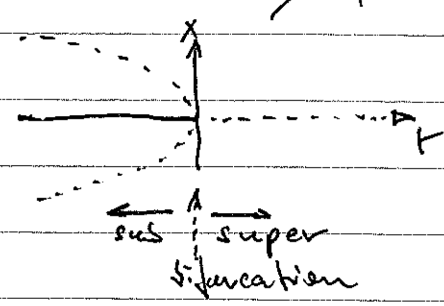
Example: Neuron response function ∝ tanh x



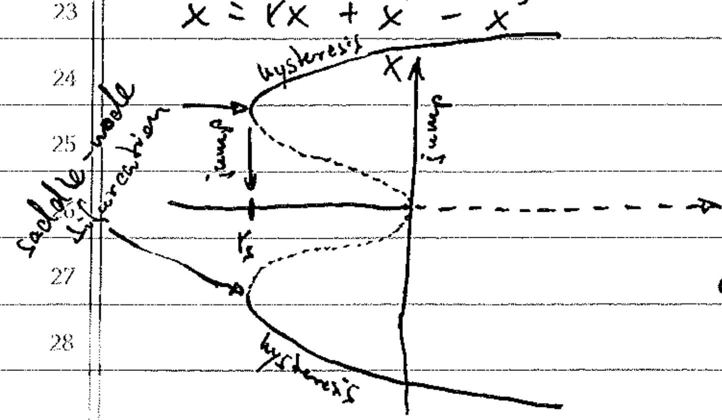
B) $\dot{x} = rx + x^3$ (subcritical pitchfork)

⇒ $x(t) \to \infty$ "blow-up"

⇒ include a "stabilizing term"



$\dot{x} = rx + x^3 - x^5$



- co-existance of 3 ($r_s < r < 0$) or 2 ($r>0$) stable fixed points. x_0 determines where system ends up as $t \to \infty$.
- changing r can cause "jumps" and "hysteresis"

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Biological application: Insect outbreak

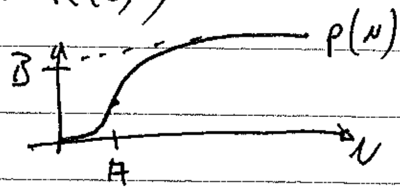
Canadian budworms attack leaves of the balsam tree and can kill entire forest in 4 years. Budworm population lives on fast time scale (months), trees grow and die on slow time scale (years).

$\Rightarrow$ To describe budworm dynamics, forest variables are constant. When forest variables drift slowly, outbreak is triggered:

- budworm population dynamics: $\dot{N} = \underbrace{RN\left(1-\frac{N}{K}\right)}_{\text{logistic growth}} - \underset{\substack{\uparrow \\ \text{predator}}}{p(N)}$

- carrying capacity K depends on forest's foliage and will be a drifting parameter in the end ($K \rightarrow K(t)$)

- budworm death due to bird predation



$$p(N) = \frac{BN^2}{A^2+N^2} \quad \text{(Hill function)}$$

- "outbreak" consists of parameter drift that leads to "jump" from low to high population levels. But what is "low" or "high"?

$\rightarrow$ dimensionless variables first: $x = \frac{N}{A}$; $\tau = \frac{Bt}{A}$; $\underset{\substack{\uparrow \\ \text{growth rate}}}{r} = \frac{RA}{B}$; $\underset{\substack{\uparrow \\ \text{carrying capacity}}}{k} = \frac{K}{A}$

$$\Rightarrow \quad \dot{x} = rx\left(1-\frac{x}{k}\right) - \frac{x^2}{1+x^2}$$

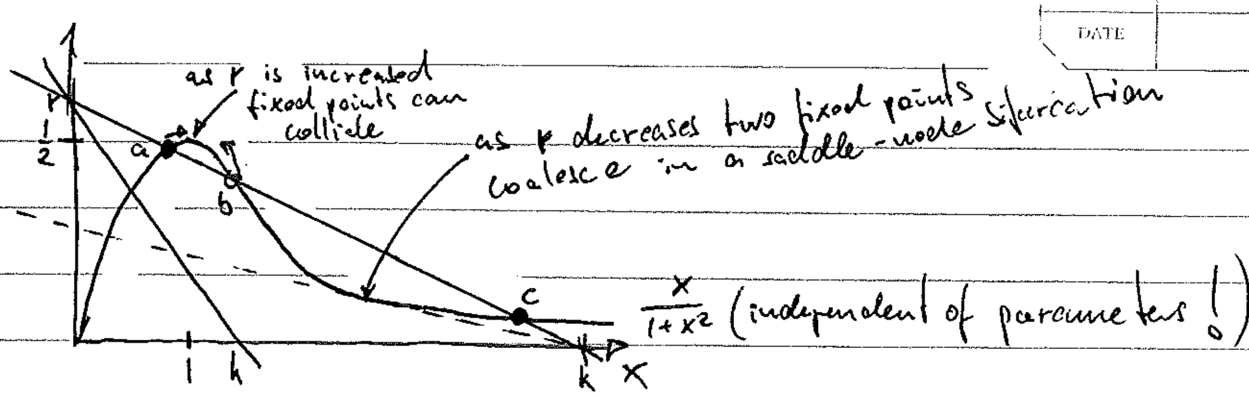
fixed points: • $x_{st1} = 0 \qquad \left.\frac{d\dot{x}}{dx}\right|_{x_{st1}} = r > 0 \Rightarrow$ unstable

(for small x predation is weak $\Rightarrow$ exponential growth)

- other fixed points are at $r\left(1-\frac{x}{k}\right) = \frac{x}{1+x^2}$

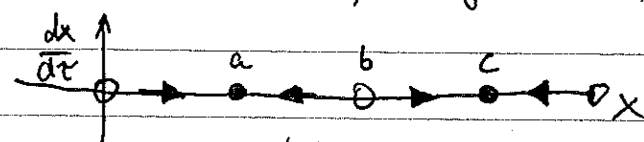
  $\rightarrow$ solve graphically

- stability: $\frac{d\dot{x}}{dx} = r - 2x\frac{r}{k} - \frac{2x}{(1+x^2)^2}$



small $k \Rightarrow$ 1 intersection ; large k 1, 2, or 3 possible fixed points

stability :



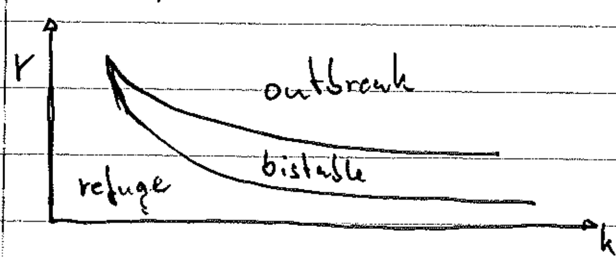
- $x_{st} = 0$ is unstable and fixed point stability has to alternate.
- small stable fixed point a (small attractive basin) is called "refuge"; large stable f.p. is the outbreak.
- outbreak occurs when $x_0 > b$; hence b is a threshold.
- outbreak can also occur when r or k change such that a disappears (saddle-node bifurcation) and the population jumps to c.

Jumps occur where curves intersect tangentially :

$$r\left(1-\frac{x}{k}\right) = \frac{x}{1+x^2} \quad \text{and} \quad \frac{d}{dx}\left[r\left(1-\frac{x}{k}\right)\right] = \frac{d}{dx}\left[\frac{x}{1+x^2}\right] \quad \text{must be fulfilled.}$$

$$\Rightarrow r(x) = \frac{2x^3}{(1+x^2)^2} \quad \text{and} \quad k(x) = \frac{2x^3}{x^2-1} \quad (\text{and as } k>0, \Rightarrow x>1)$$

$r(x)$ and $k(x)$ are called "bifurcation curves"



If S is average tree size (or foliage) then $K = K'S$ and $A = A'S$.

capacity per unit of foliage

budworms per unit of foliage

$$\Rightarrow r = \frac{RA'}{B}S \quad \text{and} \quad k = \frac{K'}{A'}$$

As forest grows r increases toward outbreak.