PREPARED BY

DATE 11/10

Linear Dynamics I

Intro example: Bacteria growing on medium

→ how many bacteria after time t?

x_t = number of baterin at time t
T = doubling time
$\Big\} \; x_{t+T} \approx 2x_t$

$$\Delta x = x_{t+T} - x_t \approx x_t$$

$$\boxed{\dot{x} = \alpha x} \qquad \dot{x} = \frac{dx}{dt} \approx \frac{\Delta x}{\Delta T} = \frac{x}{T}$$

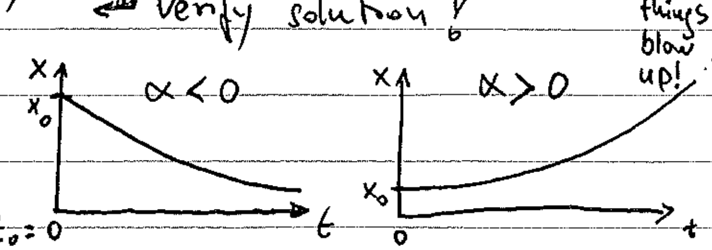
Understanding this equation (plus eigenvectors/eigenvalues) lets you understand ALL of N-dimensional linear differential equs!

x is a function of time that determines STATE of system

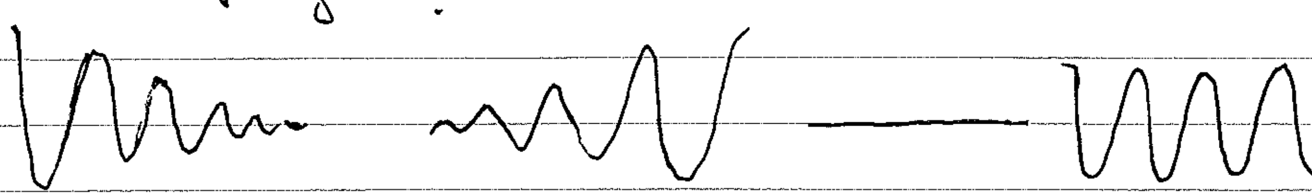
1-D solution: $$\frac{dx}{dt} = \alpha x \iff \int_{x_0}^{x} \frac{dx}{x} = \int_{t_0}^{t} \alpha\, dt \iff \ln x = \alpha t$$

$$\Rightarrow x(t) = x_0 e^{\alpha(t-t_0)}$$ ← $\frac{dx}{dt}$ verify solution!

Parameter analysis for α:



What are the following α's?



$\Rightarrow \alpha$ can be pos/neg, real/imaginary, or zero!

Example: Damped oscillations → $\alpha = -\alpha_{RE} + i\alpha_{IM}$

$$\Rightarrow x(t) = \underset{\text{Amplitude}}{x_0} \underbrace{e^{-\alpha_{RE} t}}_{\text{damping}} \cdot \underbrace{e^{i\alpha_{IM} t}}_{\text{oscillations with freq. } \alpha_{IM}}$$

Remember:
$$e^{i\alpha} = \cos\alpha + i \sin\alpha$$

Numerical Solution:

Taylor expansion

for small Δt : $x(t+\Delta t) = x + \frac{dx}{dt}\Delta t + \frac{1}{2}\frac{d^2x}{dt^2}\Delta t^2$

(neglect, Δt small!)

if we know state and its derivative we know where to go next

(approximating $x(t)$ to first-order in t)

$$\dot{x} = \frac{dx}{dt} = \alpha x \Rightarrow x(t+\Delta t) = x + \alpha x \Delta t = (1+\alpha\Delta t)\,x$$

"Euler's Method"

$$\begin{cases} x(0+\Delta t) = (1+\alpha\Delta t)\,x(0) \\ x(0+2\Delta t) = (1+\alpha\Delta t)\,x(\Delta t) = (1+\alpha\Delta t)^2 x(0) \\ \vdots \\ x(0+N\Delta t) = (1+\alpha\Delta t)^N x(0) \end{cases}$$

not very precise + stable
but Matlab can do it better.

Example : Production and Degradation of a gene product

$$(\text{constant production}) \xrightarrow{s} \overset{x}{O} \xrightarrow{k} (\text{first order degradation})$$

$$\frac{dx}{dt} = \text{production} - \text{degradation} = s - kx$$

parameters state

solution for $\dot{x} = -kx$ is $x(t) = A e^{-kt}$

Solution for $\dot{x} = s - kx$:

Ansatz $x(t) = B + Ae^{-kt}$

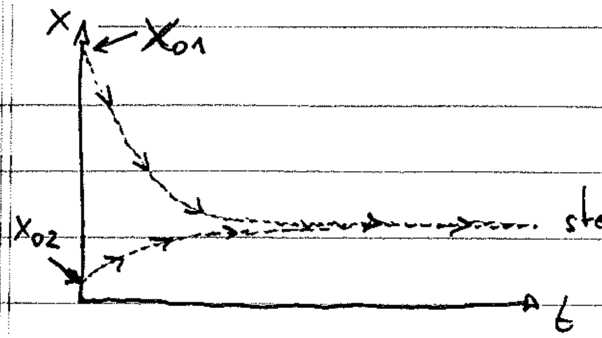
substitute $-kAe^{-kt} = s - kB - kAe^{-kt} \Rightarrow B = \frac{s}{k}$

Solution : $x(t) = \frac{s}{k} + Ae^{-kt}$ $\quad x(0) = x_0 = \frac{s}{k} + A \Rightarrow A = x_0 - \frac{s}{k}$

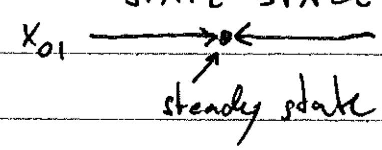
initial condition

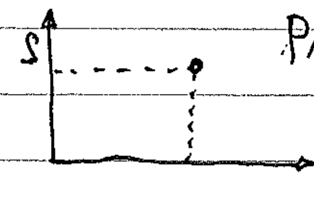
$$\Rightarrow x(t) = \frac{s}{k} + \left(x_0 - \frac{s}{k}\right)e^{-kt}$$

for $t \to \infty$: $x(t) = \frac{s}{k} \Rightarrow$ **steady state** $\Leftrightarrow \frac{dx}{dt} = 0$



STATE SPACE

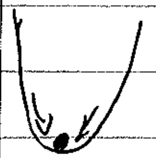




PARAMETER SPACE

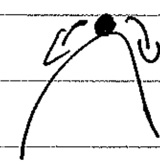
Stability analysis:

steady states can be stable or unstable (Examples?)



stable: any sufficiently small perturbation returns to steady state

unstable: some perturbations do not return



$$\dot{x} = \frac{dx}{dt} = \alpha x$$

$$\downarrow$$

$x_{st} = 0$

$\dot{x}_{st} = 0$

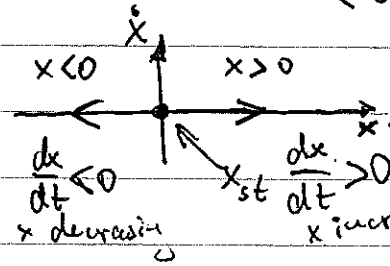
1) find steady state $x = x_{st}$ via $\left.\frac{dx}{dt}\right|_{x=x_{st}} = 0$

2) calculate derivatives $\left.\frac{d}{dx}\dot{x}\right|_{x=x_{st}}$

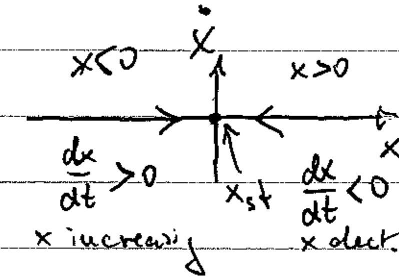
$\rightarrow > 0$ unstable

$\rightarrow < 0$ stable

$\frac{d}{dx}(\dot{x}) = \alpha > 0$:



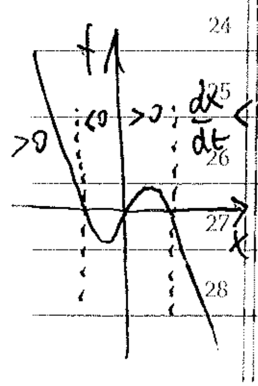
$\frac{d}{dx}(\dot{x}) = \alpha < 0$:



Linearization theorem: The dynamics of $\frac{dx}{dt} = f(x)$ is qualitatively similar to that of its linearization:

$$\frac{dx}{dt} = \left[\left.\frac{df}{dx}\right|_{x=x_{st}}\right] \cdot x$$

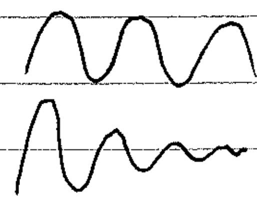
in the local vicinity of a steady state x_{st}.



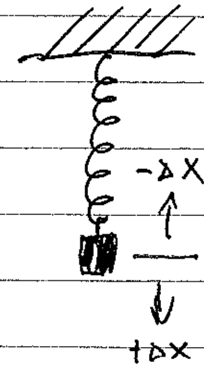
Example: $\frac{dx}{dt} = f(x) = x - x^3$ 1) $\frac{dx}{dt} = 0 \Rightarrow 3$ "fixed points" $x_{st} = \{0, 1, -1\}$

2) $\frac{df}{dx} = 1 - 3x^2 \Rightarrow f'(\pm 1) = -2 < 0$ stable $f'(0) = 1 > 0$ unstable

ODE's in 2-dimensions:



let's look at these dynamics under different light →



acceleration, position

→ $\ddot{x} = -\alpha x$

2nd derivative

define vector $\vec{x} = \begin{pmatrix} \dot{x} \\ x \end{pmatrix}$ then $\dot{\vec{x}} = \begin{pmatrix} \ddot{x} \\ \dot{x} \end{pmatrix} = \begin{pmatrix} -\alpha x \\ \dot{x} \end{pmatrix}$ or $\frac{d\vec{x}}{dt} = f(\vec{x})$

Looks familiar! using Euler we can integrate vectors:

$$\vec{x}(t+\Delta t) = \vec{x}(t) + \frac{d\vec{x}}{dt}\Delta t$$

⇒ we can write any N^{th} order dynamics as N first order ones

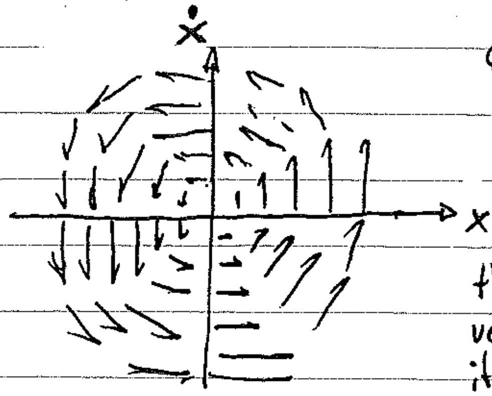
Example: $\dddot{x} = -\dot{x} + x$ → $\vec{x} = \begin{pmatrix} \ddot{x} \\ \dot{x} \\ x \end{pmatrix}$; $\dot{\vec{x}} = \begin{pmatrix} -\dot{x} + x \\ \ddot{x} \\ \dot{x} \end{pmatrix}$

Analytic solution of $\ddot{x} = -\alpha x$ is $x(t) = x_0 \sin(\sqrt{\alpha}\, t)$ ✓

check: $\dot{x}(t) = x_0 \sqrt{\alpha} \cos(\sqrt{\alpha}\, t)$

$\ddot{x}(t) = -x_0 \sqrt{\alpha}^2 \sin(\sqrt{\alpha}\, t) = -\alpha x(t)$

State Space

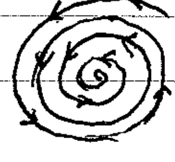


the base of each vector is at $\vec{x} = \begin{pmatrix} \dot{x} \\ x \end{pmatrix}$ and it points in direction $\frac{d\vec{x}}{dt}$

What if accel. also depends on speed $\dot{x}$? $\ddot{x} = -\gamma \dot{x} - \alpha x$

⇒ $\dot{\vec{x}} = \begin{pmatrix} \ddot{x} \\ \dot{x} \end{pmatrix} = \begin{pmatrix} -\gamma \dot{x} - \alpha x \\ \dot{x} \end{pmatrix}$ apply Euler $\vec{x}(t+\Delta t) = \vec{x}(t) + \frac{d\vec{x}}{dt}\Delta t$

$\gamma > 0$ ⇒ damped oscillations



$\gamma < 0$ ⇒ unstable oscillation