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Linear Algebra II

Change of basis and equivalence

- linear transf. are independent of basis (i.e. rot, refl., dil no basis)
- change of basis changes matrix but lin. transf. stays the same
- all matrices that represent the <u>same</u> lin. transf. are <u>equivalent</u>

$F: V \to V$ linear transf. F on v-space V

basis sets $B = \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n\}$ and $C = \{\vec{c}_1, \vec{c}_2, \dots, \vec{c}_n\}$

What is relation between F_B and F_C matrices?

$$\begin{tikzcd} V_B \arrow[r, "F"] \arrow[d, "I"'] & V_B \arrow[d, "I"] \\ V_C \arrow[r, "F"] & V_C \end{tikzcd}$$

← identity acts on V

$$F_B = I_{B\to C} \, F_C \, I_{C\to B} \qquad I_{B\to C} = \left(\vec{b}_1\big|_C, \vec{b}_2\big|_C \dots, \vec{b}_n\big|_C \right)$$

$$I_{C\to B} = I_{B\to C}^{-1} \quad \begin{pmatrix} \text{no change in} \\ V_B \to V_C \to V_B \end{pmatrix}$$

columns are components of transformed basis vectors w.r.t. C

⇒ any matrix A represents the same linear transf. as TAT^{-1} in new basis, where T corresponds to change of basis matrix ⟺ $\det T \neq 0$

⇒ $\det TAT^{-1} = \det A$

- with all this freedom of choice, what is the most simple one?

⇒ diagonal one

When is a matrix diagonalizable?

Eigenvalues / Eigenvectors

- find T such that $AT^{-1} = T^{-1}D$ where D is diagonal

$\Rightarrow A(T^{-1})_{j\cdot col} = d_i (T^{-1})_{j\cdot col}$ (just look at j^{th} column)

$\Rightarrow$ columns of T^{-1} are vectors that are unchanged by A (direction

$\Rightarrow A\vec{x} = d\vec{x}$ $\quad \vec{x} \equiv$ eigenvector $\quad d \equiv$ eigenvalue

- A is diagonalizable if exist a basis of eigenvectors
- eigenvectors provide columns for change of basis matrix (canonical $\to$ eigen)
- characteristic equation:

$$\det(\underbrace{A - dI}_{\text{not invertible}}) \overset{!}{=} 0$$

In 2d: $\det\begin{pmatrix} a-d & b \\ c & d-d \end{pmatrix} = d^2 - (a+d)d + (ad - bc) = d^2 - \mathrm{Tr}\, d + \det$

$\Rightarrow$ trace and determinant uniquely determine $d_{1/2}$ in 2d

$$d_{1/2} = \frac{\mathrm{Tr}A \pm \sqrt{(\mathrm{Tr}A)^2 - 4\det A}}{2} \equiv \frac{\mathrm{Tr}A \pm \sqrt{\Delta A}}{2}$$

$\Delta A < 0 \Rightarrow$ no eigenvectors (d_i imaginary)

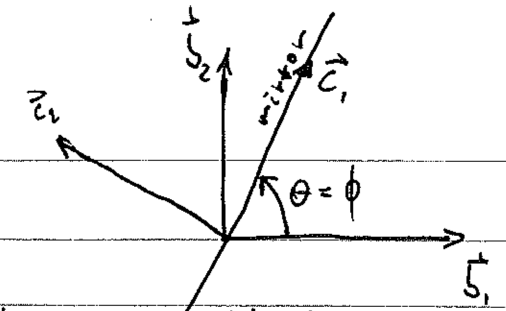
<u>example</u>: rotation matrix $\Delta A = 4(\cos^2\theta - 1)$ $\quad \theta = \begin{cases} 0 : I \\ \pi : D_{-1} \end{cases}$

$\hookrightarrow$ no vector continues to point in same direction after rotation.

$\Delta A > 0 \Rightarrow$ can be diagonalized and has eigen-basis

<u>example</u>: reflections

d_i real



<u>Reflection:</u>

→ two vectors are left fixed by reflection: $\vec{c}_1$ and $\vec{c}_2$

→ $\lambda_1 = 1 \quad \lambda_2 = -1$

→ from $Ref_\phi \Rightarrow Tr\,A = 0$; $\det A = -1 \Rightarrow \Delta A = 4$

$$\Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = T \, Ref_\phi \, T^{-1} = \underbrace{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}}_{Rot_\theta^{-1}} \begin{pmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{pmatrix} \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{Rot_\theta}$$

$\Delta A = 0 \Rightarrow \lambda_1 = \lambda_2$ real

example: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \lambda_1 = \lambda_2 = 1 \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

example: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \lambda_1 = \lambda_2 = 1 \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ no $\vec{v}_2$ ...

$$\Rightarrow 3 \text{ classes for } 2\times 2 \text{ matrices}: \quad \underbrace{\begin{pmatrix} a & -b \\ b & a \end{pmatrix}}_{\Delta A < 0}; \quad \underbrace{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}}_{>0}; \quad \underbrace{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}}_{=0}$$

<u>Exercise:</u>

1) if $\vec{v}$ is eigenvector of M is $\alpha\vec{v}$ an eigenvector of M also?

$$M\vec{v} = \lambda\vec{v} \Rightarrow M(\alpha\vec{v}) = \alpha(M\vec{v}) = (\alpha\lambda)\vec{v}$$

2) show $\det M = \prod_{i=1}^{n} \lambda_i$: $\quad D = TMT^{-1} \Rightarrow \underbrace{\det D}_{\prod \lambda_i} = \det M \det TT^{-1}$

3) what is the n^{th} root of matrix X?

$X^{\frac{1}{n}} = Y$? $\qquad PXP^{-1} = D$

$Y \equiv P^{-1} D^{1/n} P$

$\Rightarrow Y^n = P^{-1} D P = X \Rightarrow Y$ is n^{th} root

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Biological example: Red Blood Cell production