PREPARED BY	1026
DATE	11/09

Linear Algebra I

Motivation:
- gene regulatory network for development (wireing diagramm of pathways)
- cellular or organismal network
- neural network
- metabolic network

⇒ large + complex systems that are difficult to study by looking at individual component

⇒ look at system as a whole

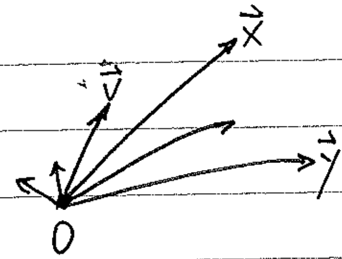
⇒ use linear algebra to facilitate compact mathematics

1). Describe "state" of system by a vector.

2). Describe "transformations" of state by matrices.

3). Find "main axes" of network by computing eigenvectors.

Vectors

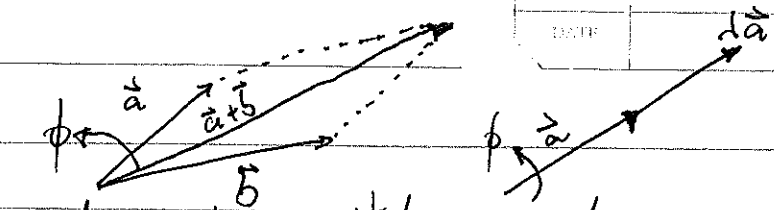


- a line segment with specified direction $\vec{x}$
- vector space: collection of all vectors from origin
- rules: $(\lambda\mu)\vec{x} = \lambda(\mu\vec{x})$ or $\lambda(\vec{x}+\vec{y}) = \lambda\vec{x}+\lambda\vec{y}$
- transformations: $F : \vec{x} \longmapsto F(\vec{x})$
- linear transformation: 1. $F(\vec{x}+\vec{y}) = F(\vec{x}) + F(\vec{y})$

  2. $F(\lambda\vec{x}) = \lambda F(\vec{x})$

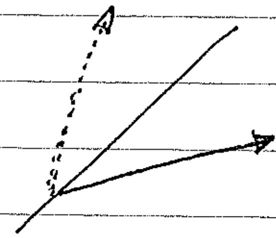
⇒ see list

Example: Rotation



1. adding first/rotating 2^{nd} = rotating 1^{st}/adding 2^{nd}
2. stretching first/rotating 2^{nd} = rotating 1^{st}/stretching 2^{nd}

Example: Reflection



= rotation in 3D of 180°

Example: Dilation

$$D_\alpha(\vec x) = \alpha\vec x$$

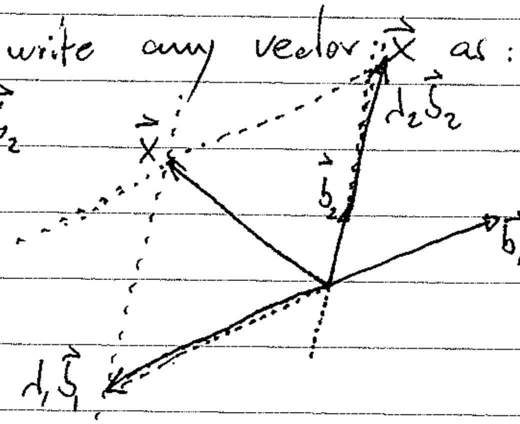
check: 1. $D_\alpha(\vec x+\vec y) = \alpha(\vec x+\vec y) = \alpha\vec x + \alpha\vec y = D_\alpha(\vec x) + D_\alpha(\vec y)$ ✓

2. $D_\alpha(\lambda\vec x) = \alpha(\lambda\vec x) = (\alpha\cdot\lambda)\vec x = \lambda(\alpha\vec x) = \lambda D_\alpha(\vec x)$ ✓

Coordinates and bases

- chose N vectors as basis vectors for n-dimensional vector space, i.e. $\vec b_1$ and $\vec b_2$ for 2d such that $\vec b_1 \neq \lambda\vec b_2$
- use "parallelogram rule" to write any vector $\vec x$ as:

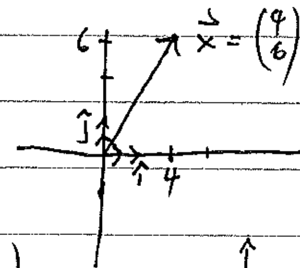
$$\vec x = \lambda_1\vec b_1 + \lambda_2\vec b_2$$



- in n-dim: $\vec x = \lambda_1\vec b_1 + \lambda_2\vec b_2 + \dots = \sum_{i=1}^{n} \lambda_i \vec b_i$ or $\begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\\ \vdots\\ \lambda_n\end{pmatrix}$
- representation of vector depends on choice of basis.

- in $\vec{b}_1, \vec{b}_2 \ldots$ basis, basis vectors are (also called $\hat{i}, \hat{j}, \hat{k}, \ldots$)

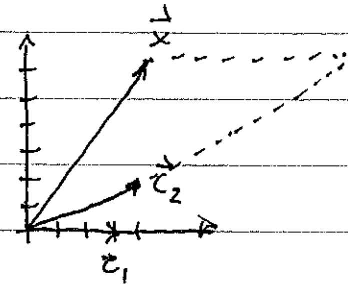
$$\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}, \begin{pmatrix}0\\1\\\vdots\\0\end{pmatrix}, \ldots, \begin{pmatrix}0\\0\\\vdots\\1\end{pmatrix}$$



<u>Example</u>: $\vec{x} = \begin{pmatrix}4\\6\end{pmatrix} = 4\hat{i} + 6\hat{j}$

new basis: $\vec{c}_1 = \begin{pmatrix}3\\0\end{pmatrix} \quad \vec{c}_2 = \begin{pmatrix}4\\2\end{pmatrix}$

$$\Rightarrow \vec{x} = \vec{c}_1\left(-\frac{8}{3}\right) + \vec{c}_2(3)$$



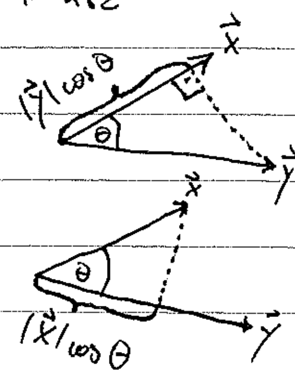
- special basis: orthonormal $\hat{i}, \hat{j}, \hat{k} \ldots$

- Operations: (addition, scalar multiplication) <u>scalar product</u>:

$$\vec{x}^T \circ \vec{y} \equiv |\vec{x}||\vec{y}| \cos\theta$$

lengths of vectors $|\vec{x}| = \sqrt{x_1^2 + x_2^2 + \ldots + x_N^2} = \vec{x}^T \circ \vec{x}$

- $\vec{x} \perp \vec{y} \Rightarrow \vec{x}^T \circ \vec{y} = 0$ and in orthonormal basis $\vec{x}^T \circ \vec{y} = \sum x_i y_i$
- $\vec{x}^T \circ (\vec{y} + \vec{z}) = \vec{x}^T \circ \vec{y} + \vec{x}^T \circ \vec{z}$
- projections:



geometrical interpretation

→ do something interesting with "states"/vectors

$\Rightarrow$ apply transformations $\Rightarrow$ matrices

Matrices (properties)

- a matrix is any rectangular array of numbers
- if n rows and m columns then called $n \times m$ matrix M (or M_{ij})
- $\vec{x}^T$ is a $1 \times m$ matrix $(\cdots)$
- $\vec{x}$ is a $m \times 1$ matrix $\begin{pmatrix} \vdots \end{pmatrix}$
- linear transformation rules apply : $(A+B)_{ij} = A_{ij} + B_{ij}$

$$(\lambda A)_{ij} = \lambda A_{ij}$$

⇒ see list

- multiplication : only multiply two matrices A and B if

$$\mathrm{col}(A) = \mathrm{row}(B)$$

example: scalar product $\vec{x}^T \circ \vec{y}$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} -2 & 2 \\ 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 5 \\ 1 & 5 \end{pmatrix}$$

not commutative

$$\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 5 & 5 \\ 5 & -5 \end{pmatrix}$$ but $$\begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 7 \\ 7 & -1 \end{pmatrix}$$

- inverse matrix : $M^{-1} \cdot M = I = MM^{-1}$ $\quad I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{pmatrix}$

inverse exists $\Longleftrightarrow$ $\det M \neq 0$

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} \qquad \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ad - cd)$$

$$M^{-1} = \frac{1}{\det M} \mathrm{Adj}\, M$$ adjugate of M

det of a $(n-1) \times (n-1)$ submatrix

$$(\mathrm{adj}\, A)_{ij} = (-1)^{i+j} \det(A_{ij})$$

Matrices as linear transformations

– a matrix takes a vector and transforms it into another vector

$$F : \vec{x} \longmapsto F(\vec{x}) = \alpha_1 F(\vec{b}_1) + \alpha_2 F(\vec{b}_2) + \dots + \alpha_n F(\vec{b}_n)$$

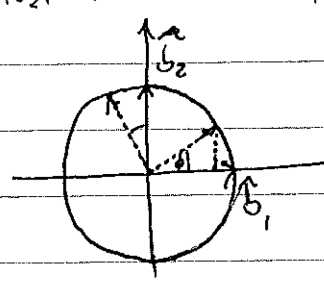
⇒ matrix $F_{\{b_i\}}$ is $n \times n$ matrix whose columns are given by n $n\times 1$ matrices of the transformation of basis vectors

$$F_{\{b_i\}}(\vec{x}) = \begin{pmatrix} F\left(\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}\right) & F\left(\begin{pmatrix}0\\1\\\vdots\\0\end{pmatrix}\right) & \dots & F\left(\begin{pmatrix}0\\0\\\vdots\\1\end{pmatrix}\right)\end{pmatrix}\begin{pmatrix}\alpha_1\\\alpha_2\\\vdots\\\alpha_n\end{pmatrix}$$

examples: 1) Dilation $D_\alpha = \alpha I_n$ $\quad (\vec{v} \mapsto \alpha\vec{v})$

2) Rotation through ϕ counterclockwise (Cartesian basis)

chose $|\vec{b}_1| = |\vec{b}_2| = 1$ and $\vec{b}_1 \perp \vec{b}_2$

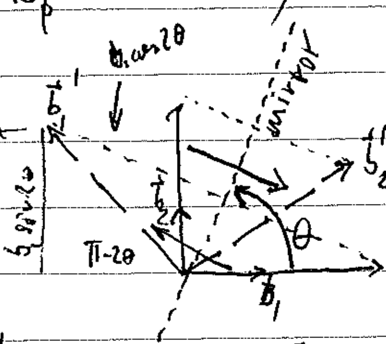


$$\begin{pmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{pmatrix} = Rot_\phi$$

(check: 1) $Rot_\pi = D_{-1} = -I_2$
2) $Rot_\phi^T = Rot_\phi^{-1}$)

3) Reflection

Ref_θ



$$Ref_\theta = \begin{pmatrix}\cos 2\theta & \sin 2\theta\\ \sin 2\theta & -\cos 2\theta\end{pmatrix}$$

$\det Ref_\theta = -1$

check: $Ref_\theta^{-1} = Ref_\theta = Ref_\theta^T$

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Exercise:

1) show that scalar product is invariant under rotations

$$\vec{x}^T \circ \vec{y} = d \Rightarrow (R_\phi \vec{x})^T \circ R_\phi \vec{y} = \vec{x}^T R_\phi^T \circ R_\phi \vec{y} = \vec{x}^T R_\phi^{-1} R_\phi \vec{y} = d$$

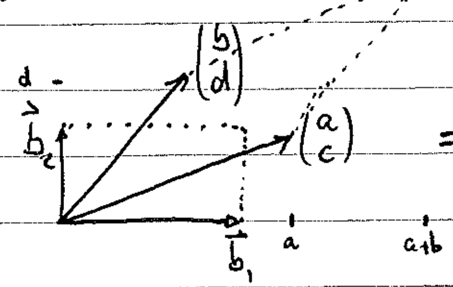
2) if $\vec{y} = M\vec{x}$ and $\vec{x}' = K\vec{x}$, $\vec{y}' = K\vec{y}$? what is M': $\vec{y}' = M'\vec{x}'$

$$\vec{y}' = K\vec{y} = KM\vec{x} = KMK^{-1}\vec{x}' = M'\vec{x}'$$

3) $M = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ what is M^ϕ when space is rotated by ϕ ?

Determinants: interpretation as linear transformation

$\rightarrow$ det = corresponds to change in volume caused by transf.



$\Rightarrow T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ transforms $\vec{b}_1, \vec{b}_2$

what about areas?

$$\text{Area} = (a|\vec{b}_1| + |\vec{b}_1|b)(c|\vec{b}_2| + d|\vec{b}_2|) - 2b|\vec{b}_1|c|\vec{b}_2| - a|\vec{b}_1|c|\vec{b}_2| - b|\vec{b}_1|d|\vec{b}_2| = (ad - bc)|\vec{b}_1||\vec{b}_2|$$

$= \det T$

<u>det</u> = volume of parallelep. formed by matrix' columns / parallelep. of basis

Also $\det(A \cdot B) = (\det A)(\det B)$