Partial Differential Equations
An Introduction

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1 Introduction

A partial differential equation (PDE) is an equation of a function of 2 or more variables, involving 2 or more partial derivatives in different variables. Unlike the case for ordinary differential equations, (ODE's) there is no complete theory for solving a general PDE – equations must be solved "one at a time."

Although there is no general method for solving an arbitrary PDE, there are several "canonical" equations which arise in many different contexts and have been studied extensively. We will look at three canonical equations which are ubiquitous in mathematics:

1) The Heat Equation:
$$\frac{\partial u}{\partial t} - k\,\nabla^2 u = 0$$

2) The Wave Equation:
$$\frac{\partial^2 u}{\partial t^2} - c^2\,\nabla^2 u = 0$$

3) The Laplace Equation:
$$\nabla^2 u = 0$$

Notation:
$$\frac{\partial u}{\partial x} = \partial_x u = u_x$$

$$\nabla^2 = \triangle = \nabla \cdot \nabla = \begin{cases} \partial_x^2 & 1-D \\ \partial_x^2 + \partial_y^2 & 2-D \\ \partial_x^2 + \partial_y^2 + \partial_z^2 & 3-D \end{cases}$$

2 The Heat Equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad \text{(one dimension)} \tag{1}$$

$$\frac{\partial u}{\partial t} = k\,\nabla^2 u \quad \text{(multi-dimensional)} \tag{2}$$

The heat equation (2) governs the dispersal of heat or diffusion of a substance. It is also sometimes called the diffusion equation.

2.1 Derivation of the Heat Equation

We will derive the 1-D heat equation $u_t - k\,u_{xx} = 0$. Derivation of the equation in more dimensions is analogous. Consider a long, thin tube filled with water, with a concentration of dye present in the water. If the tube is much longer than the diameter of the tube, we may approximate the tube as one-dimensional. We model the diffusion of the dye through the tube. Let $u(x,t)$ be the concentration of the dye in the tube at position x at time t. In the section of the tube from x_0 to x_1, the mass $M(t)$ of dye at time t is (see figure 1)

$$M(t) = \int_{x_0}^{x_1} u(x,t)dx, \quad \text{so} \quad \frac{dM}{dt} = \int_{x_0}^{x_1} u_t(x,t)dx.$$

The mass in this section of the tube cannot change except by flowing in or out of its ends. *Fick's Law* says that the rate of change of the total mass is the difference between the flow in and the flow out of the section of the tube:
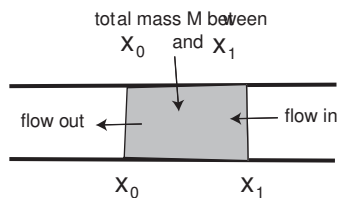

Figure 1: Flow in a tube.

The flow of dye across a given point in the tube is just the change in concentration across that point, or in other words, the x derivative of the concentration. In terms of the concentration, then,

$$\frac{dM}{dt} = \text{ flow in } - \text{ flow out } = k\,u_x(x_1,t) - k\,u_x(x_0,t),$$

where k is a proportionality constant. In the case where we are considering heat, instead of concentration, we can think of heat as flowing in and out of a section of the tube. The previous two equations are equal:

$$\int_{x_0}^{x_1} u_t(x,t)dx = ku_x(x_1,t) - ku_x(x_0,t).$$

Differentiating with respect to x_1, we get

$$u_t = k\,u_{xx},$$

the heat equation.

2.2 Solving the Heat Equation – Separation of Variables

The heat equation (1) is *linear* and *homogeneous* (there are only terms involving u and its derivatives). This means that if $u_1(x,t)$ and $u_2(x,t)$ are two solutions of (1), then

$$c_1\,u_1 + c_2\,u_2$$

is also a solution for any constants c_1 and c_2. We will find a general solution to a particular problem by adding up infinitely many solutions. This is a favorite mathematician's trick: reduce a hard problem to a series of easy problems.

To derive the equation, suppose the ends of the tube of water with some dye in it are placed in a large bath of water, so that the concentrations of the dye at the ends of the tube are held essentially constant. We may assume the constant is zero by setting $v = u - a$ and solving for v, where a is the constant concentration at the ends. Let the initial concentration of dye in the tube be described by a function $\phi(x)$ which tells us the concentration at any point x. To make things simple, let's assume that the ends of the tube are $x = 0$ and $x = 1$. (This assumption can, in fact, always be made, since we can re-scale if necessary.) Then the following initial/boundary value problem governs the diffusion of the dye in the tube:

$$\begin{cases} u_t - k\,u_{xx} = 0 & 0 < x < 1 \\ u(0,t) = u(1,t) = 0 & \text{(``Dirichlet'' boundary conditions)} \\ u(x,0) = \phi(x) \end{cases} \qquad (3)$$

(Note: This equation also governs the temperature u of a fluid in a tube with initial temperature $\phi(x)$, where the ends of the tube are held at temperature 0.)

We look for a solution of (3) of the form

$$u(x,t) = X(x)T(t)$$

where X is a function that depends only on x and T is a function that depends only on t. This is called *separation of variables* because we are separating out the x and t dependencies. There is no guarantee that this will work for any given equation, but it is often a good thing to try. Substituting this into (3),

$$XT' - k\,X''T = 0$$

Dividing by kXT and adding $\frac{X''}{X}$ to both sides, we get

$$\frac{T'}{kT} = \frac{X''}{X}.$$

Now, notice that the LHS of the above equation depends only on t, while the RHS depends only on x. So the only way they can both be equal is if they are both constant. Let's call the constant they are both equal to $-\lambda$ (the minus sign is there just to make things easier in the end). Then we get

$$\frac{T'}{kT} = \frac{X''}{X} = -\lambda.$$

Now we actually have *two* equations:

$$T' = -\lambda k\,T, \qquad X'' + \lambda X = 0,$$
$$\text{which have solutions} \quad T(t) = A\,e^{-\lambda kt}, \qquad X(x) = B\cos(\sqrt{\lambda}x) + C\sin(\sqrt{\lambda}x).$$

A, B and C are constants of integration. Now $u(x = 0,t) = u(x = 1,t) = 0$ implies that $X(0) = X(1) = 0$, so it must be that $B = 0$, but λ can take on any of the values

$$\sqrt{\lambda} = n\pi, \quad n = 0,1,2,3,\ldots.$$

For each integer n, there is a λ that satisfies the equation. These values of λ are called the *eigenvalues*. We label them according to n as

$$\lambda = \lambda_n = n^2\pi^2.$$

Corresponding to each n, there is a solution of the X equation,

$$X_n(x) = \sin(n\pi x).$$

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(We absorb C into the other coefficients.) For each λ_n there is also the corresponding T solution:

$$T_n(x) = A_n e^{-k\,n^2\pi^2 t}$$

So what do we have? Well, for each n, $u_n = X_n T_n = A_n e^{-k\,n^2\pi^2 t} \sin(n\pi x)$ solves $u_t - k\,u_{xx} = 0$. Since the sum of solutions is another solution, the sum of all the solutions is also a solution:

$$
\begin{aligned}
u(x,t) &= \sum_{n=0}^{\infty} u_n \\
&= \sum_{n=0}^{\infty} X_n T_n \\
&= \sum_{n=0}^{\infty} A_n e^{-k\,n^2\pi^2 t} \sin(n\pi x) \quad\quad (4)
\end{aligned}
$$

This (4) is **the** solution to (3) *as long as* it satisfies the initial condition. That is, if

$$u(x,0) = \phi(x) = \sum_{n=0}^{\infty} X_n(x) T_n(0) = \sum_{n=0}^{\infty} A_n \sin(n\pi x). \quad\quad (5)$$

In other words, as long as we can find a sequence of numbers A_n that satisfies the above (5), then we have solved the equation. Actually, as long as ϕ is piecewise continuously differentiable (it has a continuous derivative on all but a finite number of points), we **can** find A_n to satisfy (5), and thus solve the heat equation (3) completely! This remarkable fact is due to what is called the "completeness" of the trigonometric functions on the space of piecewise continuously differentiable functions. Equation (5) is actually just the Fourier sine expansion of $\phi(x)$.

That's nice, but does it do us any good? In order to be useful, we must be able to actually determine the constants A_n. In fact, we can. To find A_n, multiply (5) by $\sin(m\pi x)$, then integrate from 0 to 1:

$$\int_0^1 \phi(x) \sin(m\pi x) dx = \sum_{n=0}^{\infty} A_n \int_0^1 \sin(n\pi x) \sin(m\pi x) dx$$

Now we use the fact that

$$\int_0^1 \sin(n\pi x) \sin(m\pi x) dx = \begin{cases} \frac{1}{2} \text{ if } n = m \\ 0 \text{ if } n \neq m \end{cases}$$

We then deduce that

$$A_m = 2 \int_0^1 \phi(x) \sin(m\pi x) dx,$$

which gives us the last piece of the solution to (3).

Example: Suppose $k = 1$, and the initial density is

$$\phi(x) = \sin(\pi x)$$

Examine (5) to convince yourself that $A_1 = 1$ and $A_m = 0 \; m \neq 1$. Then the solution to (3) is

$$u(x,t) = e^{-\pi^2 t} \sin(\pi x).$$

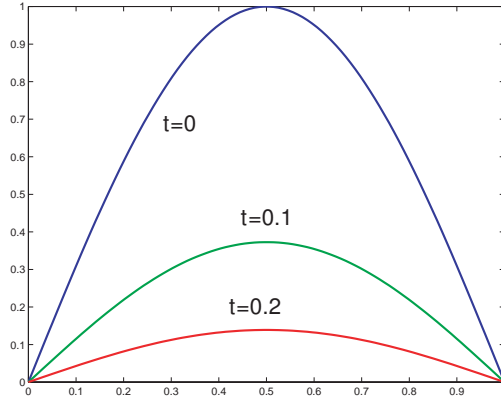The concentration of the dye (or temperature) decreases exponentially to zero, as seen in Figure 2.

Figure 2: Solution of the heat equation at various times.

2.3   Different Boundary Conditions

If in (3) we used the boundary conditions

$$u_x(0,t) = u_x(1,t) = 0,$$

called Neumann boundary conditions, we would get a cosine series instead of a sine series. The final solution (try it!) would be

$$u(x,t) = A_0 + \sum A_n e^{-n^2\pi^2 t} \cos n\pi x.$$

This corresponds to restricting the flow of dye out of the tube - $u_x = 0$ means the concentration is not changing in the x direction. Thus the concentration approaches the constant A_0 exponentially.

You can solve the equation (3) with a variety of different boundary conditions and get variations on the solution we obtained. In each case we use the orthoganality of the trig functions and the facts:

$$\int_0^1 \sin(n\pi x)\sin(m\pi x)dx = \int_0^1 \cos(n\pi x)\cos(m\pi x)dx = \begin{cases} \frac{1}{2} & m = n \\ 0 & m \neq n \end{cases}$$

$$\int_0^1 \sin(n\pi x)\cos(m\pi x)dx = 0 \quad \text{all } m,n$$

We will not look here at diffusion on the whole real line, which requires a different technique.

3   The Wave Equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \nabla^2 u = 0 \tag{6}$$

The wave equation (6) governs the propagation (you guessed it!) of waves through a medium, e.g. light, water waves, plucked strings, etc.

3.1   Derivation of the Wave Equation

We will derive the wave equation for a 1-D string undergoing small transverse vibrations in a plane. At each point x the string has an amplitude $u(x,t)$ at time t. If the string is pulled (plucked) out

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of its equilibrium shape, a tension $T(x,t)$ will be exerted at each point of the string, exerting a force on the string that is tangential to the string. The slope of the string at each point is $u_x(x,t)$. We write down Newton's law (F=ma) for the part of the string between any two points x_0 and x_1. Let ρ be the (constant) density of the string.
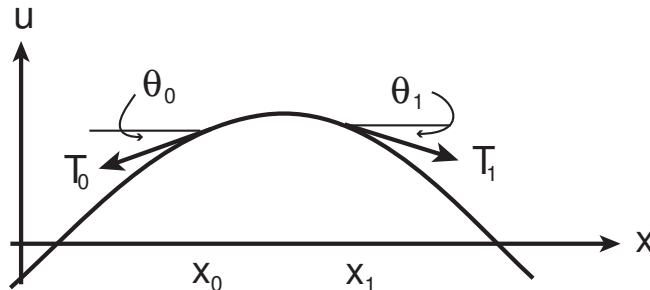

Figure 3: Tension on a string

We assume that there is no motion of the string in the x-direction. Then we can write Newton's law in its longitudinal and transverse components, respectively:

$$\underbrace{T_1 \cos\theta_1 - T_0 \cos\theta_0}_{x \text{ component of force on section of string}} = 0$$

$$\underbrace{T_1 \sin\theta_1 - T_0 \sin\theta_0}_{y \text{ component of force on section of string}} = \begin{array}{l}\text{rate of change of momentum} \\ \text{of section of string}\end{array}$$

Next we use the fact that the tangent angle θ between the tangent to the curve and x-axis is

$$\cos\theta = \frac{1}{\sqrt{1+u_x^2}}, \quad \sin\theta = \frac{u_x}{\sqrt{1+u_x^2}}.$$

Then we write Newton's laws in terms of u_x:

$$\left. \frac{T}{\sqrt{1+u_x^2}} \right|_{x_0}^{x_1} = 0$$

$$\text{and} \quad \left. \frac{Tu_x}{\sqrt{1+u_x^2}} \right|_{x_0}^{x_1} = \frac{d^2}{dt^2} \int_{x_0}^{x_1} \rho u \, dx.$$

The RHS's are the components of the mass times the acceleration integrated over the piece of string. (There is no longitudinal motion.) Now we assume that the motion is small, or in particular that $|u_x|$ is small. We then use the approximation

$$\sqrt{1+u_x^2} \approx 1.$$

We also assume that T is constant (in x and t). Differentiating the second equation with respect to x_1, we get

$$(Tu_x)_x = \rho\, u_{tt}$$

With $c = \sqrt{\frac{T}{\rho}}$ this is just

$$u_{tt} = c^2\, u_{xx},$$

the wave equation!

3.2 Solving the Wave Equation – D'Alembert's Formula

Notice that the wave equation is second order and linear. Thus we will need two auxilary (initial or boundary) conditions to specify a unique solution. Linearity and homogeneity imply that if u_1 and u_2 are two solutions of (6), then $\alpha u_1 + \beta u_2$ is also a solution for any constants α and β.

Let's examine the one-dimensional wave equation on the whole real line ($-\infty < x < \infty$). The equation then governs the amplitude of an infinitely long vibrating string. We need two initial conditions. We will take as given the profiles of the initial amplitude and velocity of the string. The problem then is to solve

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & -\infty < x < \infty \\ u(x,0) = \phi(x) & u_t(x,0) = \psi(x) \end{cases} \tag{7}$$

We write

$$u_{tt} - c^2 u_{xx} = (\partial_t - c\partial_x)(\partial_t + c\partial_x)u = 0$$

This implies that

$$u_t - cu_x = 0 \quad \text{or} \quad u_t + cu_x = 0 \tag{8}$$

These equations are examples of the "transport equation,"

$$av_t + bv_x = 0 \tag{9}$$

which governs the flow of a one-dimensional fluid. To solve (9) we notice that along the lines $x(t) = \frac{b}{a}t + x_0$, called the characteristic lines,

$$\begin{aligned} \tfrac{d}{dt}v(x(t),t) &= v_t + v_x \tfrac{dx}{dt} \\ &= v_t + v_x \tfrac{b}{a} \\ &= 0. \end{aligned}$$

so v is constant on the characteristic lines. Thus, on any of these lines v has the same value as at $x(0) = x_0$. Therefore

$$\begin{aligned} v(x,t) &= v(x(o), t=0) \\ &= v(x - \tfrac{b}{a}t, 0) \\ &= h(x - \tfrac{b}{a}t) \end{aligned}$$

is a solution for any (differentiable) function h. You can verify that this actually does solve the transport equation (9) and that if $h(x)$ is the initial condition of (9), it satisfies the initial conditions as well.

Now back to the wave equation. The two equations (8) we can now solve as we did the transport equation. We obtain respectively

$$u_1(x,t) = f(x + ct) \quad \text{and} \quad u_2(x,t) = g(x - ct)$$

as solutions to (8) for any f, g. Both of these solve the wave equation, $u_{tt} - c^2 u_{xx} = 0$, thus their sum also solves it. Thus, the general solution is

$$u(x,t) = f(x + ct) + g(x - ct) \tag{10}$$

Now to satisfy the initial conditions. We calculate

$$\begin{aligned} u(x,0) = & \phi(x) & = f(x) + g(x) \\ u_t(x,0) = & \psi(x) & = cf'(x) - cg'(x) \\ \Rightarrow & f' & = \tfrac{1}{2}(\phi' + \tfrac{1}{c}\psi) \\ \text{and} & g' & = \tfrac{1}{2}(\phi' - \tfrac{1}{c}\psi) \end{aligned}$$

Integrating, we obtain

$$f(s) = \frac{1}{2}\phi(s) + \frac{1}{2c}\int_0^s \psi(s)ds + A, \quad \text{and}$$

$$g(s) = \frac{1}{2}\phi(s) - \frac{1}{2c}\int_0^s \psi(s)ds + B$$

A and B are arbitrary, but since $\phi = f + g$, $A + B = 0$. Combining these facts we obtain the solution to the wave equation (6).

$$u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds \tag{11}$$

Equation (11) is the famous D'Alembert formula for the solution to the wave equation, due to the Frenchman, circa 1746. Notice that u is the sum of a left travelling $f(x+ct)$ and right travelling $g(x-ct)$ wave. If $\psi = 0$ then each have the same shape as the initial profile of the wave, but $1/2$ the amplitude.

3.3   The Wave Equation on a Finite Interval – Separation of Variables

Of course, there is no such thing as an infinitely long string. So let's now examine the more physically realistic situation of a string in a finite interval. Suppose we have a string (such as a guitar or violin string) clamped at it's endpoints $x = 0$ and $x = 1$. We give the string an intial profile $\phi(x)$ (a pluck) and velocity $\psi(x)$. Then the equation governing the string's amplitude is

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & 0 < x < 1 \\ u(0) = u(1) = 0 & \text{(Dirichlet boundary conditions)} \\ u(x,0) = \phi(x), \; u_t(x,0) = \psi(x) \end{cases} \tag{12}$$

As we did with the heat equation, we try the separation of variables technique and look for a solution of the form

$$u(x,t) = X(x)T(t)$$

Then

$$XT'' - c^2 X''T = 0$$

Dividing by $c^2 XT$, we get

$$\frac{T''}{c^2 T} = \frac{X''}{X} = -\lambda$$

λ must, of course, be constant. We thus arrive at the two equations:

$$T'' + c^2\lambda T = 0 \qquad\qquad X'' + \lambda X = 0$$
$$T(t) = A\cos(c\sqrt{\lambda}t) + B\sin(c\sqrt{\lambda}t) \qquad X(x) = C\cos(\sqrt{\lambda}x) + D\sin(\sqrt{\lambda}x)$$

Again, $u(0) = u(1) = 0 \Rightarrow X(0) = X(1) = 0$ which implies that $A = 0$, and

$$\sqrt{\lambda} = n\pi \quad \Rightarrow \lambda_n = n^2\pi^2$$

where n is an integer are the only allowed values of λ. Thus for each n we have the X and T solutions

$$X_n(x) = \sin(n\pi x), \;\; T_n(t) = A_n\cos(cn\pi t) + B_n\sin(cn\pi t)$$

The T_n's are the "harmonics" of the string. Each $u_n = X_n T_n$ solves the wave equation. The sum

$$
\begin{aligned}
u(x,t) &= \sum_{n=0}^{\infty} X_n(x) T_n(t) \\
&= \sum_{n=0}^{\infty} [A_n \cos(cn\pi t) + B_n \sin(cn\pi t)] \sin(n\pi x)
\end{aligned}
\tag{13}
$$

is also a solution. Equation (13) is **the** solution if it satisfies the initial conditions, *i.e.* if

$$
u(x,0) = \phi(x) = \sum_{n=0}^{\infty} X_n(x) T_n(0) = \sum_{n=0}^{\infty} A_n \sin(n\pi x)
$$

and

$$
u_t(x,0) = \psi(x) = \sum_{n=0}^{\infty} X_n T_n'(0) = \sum_{n=0}^{\infty} B_n cn\pi \sin(n\pi x)
$$

Again, as long as ϕ and ψ are piecewise continuously differentiable, then we can find A_n and B_n to satisfy these equations. As we did for the heat equation we find

$$
A_m = 2 \int_0^1 \phi(x) \sin(m\pi x) dx \quad \text{and} \quad B_m = \frac{2}{cm\pi} \int_0^1 \psi(x) \sin(m\pi x) dx
\tag{14}
$$

Thus, with (13) and (14) we have the complete solution to the wave equation (12).

Example: Suppose we have the initial conditions

$$
\phi(x) = \sin(\pi x), \quad \psi(x) = 0
$$

This corresponds to the situation where the string is plucked, but not given any initial velocity. We find that $B_m = 0$ for all m, $A_1 = 1$ and $A_m = 0$ for $m \neq 1$. Thus the solution is:

$$
u(x,t) = \sin(\pi x) \cos(c\pi t)
$$

The string vibrates in time indefinitely with frequency of oscillation $c/2$. By changing the tension in the string you change the frequency of oscillation – this is intuitively obvious, but the equation tells us quantitatively how to change the frequency.

3.4   Other Boundary Conditions

Another typical boundary condition is

$$
u_x(0,t) = u_x(1,t) = 0,
$$

the Neumann boundary condition. This corresponds to the case where the ends of the string are allowed to move along a rod, but the angle of the string is held constant at the ends. In this case we get a cosine series instead of a sine series.

4   The Laplace Equation

$$\nabla^2 u = 0 \tag{15}$$

Although a simple looking equation, the Laplace equation arises in numerous contexts in mathematics. A function which satisfies it (15) is called a **harmonic function**. In order to specify a solution to (15) in a certain domain D, we must know the conditions on the boundary of D, ∂D (the boundary conditions). Thus, the Laplace equation is usually posed as:

$$\begin{cases} \nabla^2 u = 0 & \text{in } D \\ u(\vec{x}) = h(\vec{x}) & \text{on } \partial D \end{cases} \tag{16}$$

or something like this, where h is some given function. In general, this equation is very difficult to solve analytically unless D has a high degree of symmetry, for example if D is a circle, wedge, annulus or rectangle. We will look at one example shortly.

Contexts in which Laplace's Equation Arises

1) **Complex Variables:** If $f(z) = f(x + iy) = u(x,y) + iv(x,y)$ is analytic (differentiable) then u and v are both harmonic functions. Conversely, if u is a harmonic function then there is a harmonic function v such that $f = u + iv$ is analytic. For example, $f(z) = z^2$ is analytic. $f(x + iy) = x^2 - y^2 + i2xy$, so $u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$ are both harmonic. (Check it!)

2) **Electrostatics:** From Maxwell's equations, one has $\nabla \times E = 0$ and $\nabla \cdot E = 4\pi\rho$, where ρ is the charge density. The first equation implies $E = -\nabla\phi$ for a scalar function ϕ (called the electric potential). Therefore,

$$\nabla^2 \phi = \nabla \cdot \nabla \phi = -\nabla \cdot E = -4\pi\rho.$$

If the charge density is zero (empty space) this is Laplace's equation.

3) **Steady Fluid Flow:** Assume that the flow of a fluid is irrotational (no eddies) so that $\nabla \times v = 0$, where v is the velocity of the fluid, assumed independent of time. Assume that the fluid is incompressible (e.g. water) and that there are no sources or sinks. Then $\nabla \cdot v = 0$. Hence $v = -\nabla\phi$ for some ϕ (called the velocity potential) and $\nabla^2 \phi = \nabla \cdot v = 0$, which is Laplace's equation again.

4) **Steady States:** In general, Laplace's equation describes a steady state, or ground state. For example, in the heat equation, if u is not changing in time ($\partial_t u = 0$) then we just have Laplace's equation. This means that the equilibrium state of a diffused substance is described by the Laplace equation.

Laplace's Equation in a Box

Consider the simple example of the 2-D problem in a box:

$$\begin{cases} \nabla^2 u = u_{xx} + u_{yy} = 0 & 0 < x, y < 1 \\ u(0,y) = u(1,y) = u(x,1) = 0 \\ u(x,0) = \phi(x) \end{cases} \tag{17}$$

This problem corresponds to the equilibrium temperature inside a box where all sides are held at temperature 0 except the bottom side. We try the separation of variables technique again and look for a solution of the form $u(x,y) = X(x)Y(y)$. Substituting this into (17), we get

$$X''Y + XY'' = 0$$

Dividing by XY we obtain

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

where λ is constant. We have the two equations:

$$X'' + \lambda X = 0 \qquad\qquad Y'' - \lambda Y = 0$$
$$X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x) \quad Y(y) = C\cosh(\sqrt{\lambda}y) + D\sinh(\sqrt{\lambda}y)$$

Now, $u(0,y) = u(1,y) = 0 \Rightarrow X(0) = X(1) = 0$, which implies that $A = 0$ and

$$\sqrt{\lambda} = n\pi \ \Rightarrow \lambda = \lambda_n = n^2\pi^2$$

where n is an integer are the only allowed values (the eigenvalues) of λ. For each n we have the X and Y solutions

$$X_n(x) = \sin(\sqrt{\lambda}x), \ \ Y_n(y) = C_n\cosh(n\pi y) + D_n\sinh(n\pi y)$$

Now, $u(x,1) = 0 \Rightarrow Y(1) = 0$, which implies that

$$D_n = -C_n\coth(n\pi)$$

All we need now are the C_n's. These will be determined by $\phi(x)$. We take the sum of all the solutions $u_n = X_n Y_n$ to get

$$
\begin{aligned}
u(x,t) \ &= \ \sum_{n=0}^{n=\infty} X_n(x)Y_n(y) \\
&= \ \sum_{n=0}^{n=\infty} C_n\sin(n\pi x)[\cosh(n\pi y) - \coth(n\pi)\sinh(n\pi y)] \quad (18)
\end{aligned}
$$

To find the C_n's we do as before:

$$u(x,0) = \phi(x) = \sum_{n=0}^{n=\infty} X_n(x)Y_n(0) = \sum_{n=0}^{n=\infty} C_n\sin(n\pi x)$$

Multiplying by $\sin(m\pi x)$ and integrating, we arrive at the following formula for C_m:

$$C_m = 2\int_0^1 \phi(x)\sin(m\pi x)dx \qquad\qquad (19)$$

Thus (18) and (19) furnish us with the solution of the Laplace equation (17).

In this simple case we have found a solution. However, for more general boundary conditions the separation of variables technique will not work. (Try it!) Imagine, then, trying to solve the problem in an irregularly shaped region instead of a box. In fact, this is an advanced problem for which much research has been devoted. Analytical solutions are generally not possible, so research has been mostly involved with trying to find good numerical techniques for approximating the solution.

(A few) General Properties of Harmonic Functions:

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Although we cannot in general solve the equation analytically, we can make qualitative statements about the behavior of the solution to the general Laplace equation (16). We state two of these (without proof) here:

1) **The maximum/minimum principle:** Suppose u solves (16), where D is connected, and that u is continuous on $D \cup \partial D$. Then the maximum and minimum values of u are achieved on ∂D and nowhere in D, unless u is constant.

2) **Differentiability:** Suppose u is harmonic in an open set D. Then u possesses all partial derivatives of all orders in D, *i.e.* u is infinitely differentiable in D.

3) **Mean Value Property:** (In 2-D) Let u be a harmonic function in a disk D, continuous on its closure $D \cup \partial D$. Then the value of u at the center of D equals the average of u on its circumference. If D is the disk centered at the origin, with radius a,

$$u(\vec{0}) = \frac{1}{2\pi a} \int_{|\vec{x}|=a} u(\vec{x}) ds$$

Analagous statements exist for all dimensions.

4) **Dirichlet's Principle:** If we define the energy as

$$E(u) = \frac{1}{2} \int_D |\nabla u|^2 dx,$$

then among all functions which satisfy $u(x) = h(x)$ on ∂D, the one that minimizes the energy E is the one that is harmonic in D.