MOL410/510 Problem Set 1 Solutions - Linear Algebra

1. **Matrix mutliplication.**

   - $C = AB = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 6 & 5 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 1\cdot 6 + 3\cdot 3 & 1\cdot 5 + 3\cdot 2 \\ 4\cdot 6 + 2\cdot 3 & 4\cdot 5 + 2\cdot 2 \end{pmatrix} = \begin{pmatrix} 15 & 11 \\ 30 & 24 \end{pmatrix}$

   - $D = BA = \begin{pmatrix} 6 & 5 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 6\cdot 1 + 5\cdot 4 & 6\cdot 3 + 5\cdot 2 \\ 3\cdot 1 + 2\cdot 4 & 3\cdot 3 + 2\cdot 2 \end{pmatrix} = \begin{pmatrix} 26 & 28 \\ 11 & 13 \end{pmatrix}$

2. **Matrix mutliplication.**

   - $C = AB = \begin{pmatrix} 2 & 1 \\ 4 & 3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 6 & 5 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 2\cdot 6 + 1\cdot 3 & 2\cdot 5 + 1\cdot 2 \\ 4\cdot 6 + 3\cdot 3 & 4\cdot 5 + 3\cdot 2 \\ 3\cdot 6 + 1\cdot 3 & 3\cdot 5 + 1\cdot 2 \end{pmatrix} = \begin{pmatrix} 15 & 12 \\ 33 & 26 \\ 21 & 17 \end{pmatrix}$

   - BA does not exist.

3. **Matrix mutliplication.**

   - $C = AB = \begin{pmatrix} 3 & -3 & 2 \\ 1 & 5 & -1 \\ -4 & -6 & 2 \end{pmatrix} \begin{pmatrix} 8 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 3\cdot 8 - 3\cdot 1 + 2\cdot 4 \\ 1\cdot 8 + 5\cdot 1 - 1\cdot 4 \\ -4\cdot 8 - 6\cdot 1 + 2\cdot 4 \end{pmatrix} = \begin{pmatrix} 29 \\ 9 \\ -30 \end{pmatrix}$

   - $D = B^{\mathrm{T}} A^{\mathrm{T}} = \begin{pmatrix} 8 & 1 & 4 \end{pmatrix} \begin{pmatrix} 3 & 1 & -4 \\ -3 & 5 & -6 \\ 2 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 29 & 9 & -30 \end{pmatrix}$

   D is the transpose of C.

4. **Dot Product.**

   - $\mathbf{x}^{\mathrm{T}} \cdot \mathbf{y} = 1\cdot 0 + 3\cdot 2 = 6$

   - When $\theta = 30°$, $R = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$. Therefore,

   $$\mathbf{x}' = R\mathbf{x} = \begin{pmatrix} \frac{\sqrt{3}}{2}\cdot 1 + \frac{1}{2}\cdot 3 \\ -\frac{1}{2}\cdot 1 + \frac{\sqrt{3}}{2}\cdot 3 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2}\cdot(1+\sqrt{3}) \\ \frac{1}{2}\cdot(3\sqrt{3}-1) \end{pmatrix}$$

$$\mathbf{y}' = R\mathbf{y} = \begin{pmatrix} \frac{\sqrt{3}}{2}\cdot 0 + \frac{1}{2}\cdot 2 \\ \\ -\frac{1}{2}\cdot 0 + \frac{\sqrt{3}}{2}\cdot 2 \end{pmatrix} = \begin{pmatrix} 1 \\ \\ \sqrt{3} \end{pmatrix}$$

$\mathbf{x}'^{\mathrm{T}} \cdot \mathbf{y}' = \frac{\sqrt{3}}{2}\cdot(1+\sqrt{3})\cdot 1 + \frac{1}{2}\cdot(3\sqrt{3}-1)\sqrt{3} = 6$, the same value before the rotation.

- For any θ, $\mathbf{x}' = \begin{pmatrix} \cos\theta + 3\sin\theta \\ -\sin\theta + 3\cos\theta \end{pmatrix}$ and $\mathbf{y}' = \begin{pmatrix} 2\sin\theta \\ 2\cos\theta \end{pmatrix}$.

  Therefore, $\mathbf{x}'^{\mathrm{T}} \cdot \mathbf{y}' = 2\cos\theta\sin\theta + 6\sin^2\theta - 2\cos\theta\sin\theta + 6\cos^2\theta = 6$.

5. **Dot product and a non-orthonormal base transform.**

   • To find the representation of a vector in a new basis direction, we first write the basis in terms of the old basis, with each basis written as a column in a matrix (see solution to problem 10).

$$B = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ \\ \frac{1}{\sqrt{2}} & 1 \end{pmatrix}$$

   Then to find how a matrix in the old basis is written in the new basis, find the inverse of B.

$$B^{-1} = \begin{pmatrix} \sqrt{2} & 0 \\ \\ -1 & 1 \end{pmatrix}$$

   and multiply it onto the vectors in the old basis, i.e. $\mathbf{x}'^{\mathrm{T}} = B^{-1}\mathbf{x}$ and $\mathbf{y}'^{\mathrm{T}} = B^{-1}\mathbf{y}$.

   Therefore, $\mathbf{x}'^{\mathrm{T}} = \begin{pmatrix} \sqrt{2} & 2 \end{pmatrix}$ and $\mathbf{y}'^{\mathrm{T}} = \begin{pmatrix} 0 & 2 \end{pmatrix}$.

   • $\mathbf{x}'^{\mathrm{T}} \cdot \mathbf{y}' = 4$. The dot product is not conserved because the new basis is not orthogonal. However, the vectors did not change at all, we only wrote them in a new basis. Therefore, the overlap between the two vectors did not change, i.e. it still equals 6. The dot product and the overlap between the two vectors are only the same in orthonormal basis changes.

6. **Matrix Multiplication.** • Let's say there are N time points, then:

$$\mathbf{s} = L\mathbf{v} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 1 & \dots & 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix} = \begin{pmatrix} v_1 \\ v_1 + v_2 \\ \vdots \\ \sum_{i=1}^{N} v_i \end{pmatrix}$$

7. **Matrix Multiplication.**

• $M\Lambda = \begin{pmatrix} a1 & b1 & c1 \\ a2 & b2 & c2 \\ a3 & b3 & c3 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} a1 \cdot \lambda_1 & b1 \cdot \lambda_2 & c1 \cdot \lambda_3 \\ a2 \cdot \lambda_1 & b2 \cdot \lambda_2 & c2 \cdot \lambda_3 \\ a3 \cdot \lambda_1 & b3 \cdot \lambda_2 & c3 \cdot \lambda_3 \end{pmatrix}$

Each column is the i^{th} column of M multiplied by λ_i.

• Let's first examine

$$M\mathbf{v_i} = \begin{pmatrix} \sum_{j=1}^{N} M_{1j} v_{ij} \\ \sum_{j=1}^{N} M_{2j} v_{ij} \\ \vdots \\ \sum_{j=1}^{N} M_{Nj} v_{ij} \end{pmatrix} = \lambda_i \begin{pmatrix} v_{i1} \\ v_{i2} \\ \vdots \\ v_{iN} \end{pmatrix} \equiv \lambda_i \begin{pmatrix} | \\ \mathbf{v_i} \\ | \end{pmatrix} \tag{1}$$

where v_{ij} is the j^{th} element of vector $\mathbf{v_i}$ and M_{ij} is the i^{th} row and j^{th} column of M.

Now, let's calculate

$$MV = \begin{pmatrix} \sum_{i=1}^{N} M_{1i} v_{1i} & \sum_{i=1}^{N} M_{1i} v_{2i} & \ldots & \sum_{i=1}^{N} M_{1i} v_{Ni} \\ \sum_{i=1}^{N} M_{2i} v_{1i} & \sum_{i=1}^{N} M_{2i} v_{2i} & \ldots & \sum_{i=1}^{N} M_{2i} v_{Ni} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^{N} M_{Ni} v_{1i} & \sum_{i=1}^{N} M_{Ni} v_{2i} & \ldots & \sum_{i=1}^{N} M_{Ni} v_{Ni} \end{pmatrix} \tag{2}$$

Using the result from the previous portion of the problem, we can write

$$V\Lambda = \begin{pmatrix} | & | & | \\ \lambda_1 \mathbf{v_1} & \ldots & \lambda_N \mathbf{v_N} \\ | & | & | \end{pmatrix} \tag{3}$$

Setting $MV = V\Lambda$, we see that each column of eqns. (2) and (3) satisfy eqn. (1).

8. **Inverses and matrix multiplication.**

• $AA^{-1} = \frac{1}{ad-bc}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} ad-bc & -ab+ba \\ cd-dc & -cb+da \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

• The inverse does not exist when $ad = bc$.

• Two vectors are parallel if they can be written as scalar multiples of one another. That is, if $\mathbf{u}$ is parallel to $\mathbf{v}$, then $\mathbf{u} = \alpha\mathbf{v}$, where $\alpha \neq 0$ is a scalar. If our vectors are $\begin{pmatrix} a \\ c \end{pmatrix}$ and $\begin{pmatrix} b \\ d \end{pmatrix}$, and $ad = bc$, then we can rewrite the two vectors as $\begin{pmatrix} a \\ ad/b \end{pmatrix} = a\begin{pmatrix} 1 \\ d/b \end{pmatrix}$ and $\begin{pmatrix} b \\ bc/a \end{pmatrix} = b\begin{pmatrix} 1 \\ c/a \end{pmatrix}$, but $d/b = c/a$ so that both vectors are scalar multiples of one another and are therefore parallel. A similar argument holds for the row vectors.

9. **Inverses.**

• Left multiply $(AB)^{-1} = B^{-1}A^{-1}$ on each side with (AB), then $(AB)(AB)^{-1} = I$ and $(AB)B^{-1}A^{-1} = ABB^{-1}A^{-1} = AIA^{-1} = AA^{-1} = I$.

10. **Changing bases.** Let's expand the equation:

$$\mathbf{x} = \mathbf{b_1}y_1 + \mathbf{b_2}y_2 + \mathbf{b_3}y_3 = \begin{pmatrix} b_{11}y_1 + b_{21}y_2 + b_{31}y_3 \\ b_{12}y_1 + b_{22}y_2 + b_{32}y_3 \\ b_{13}y_1 + b_{23}y_2 + b_{33}y_3 \end{pmatrix} \tag{4}$$

where b_{ij} is the j^{th} element of vector $\mathbf{b_i}$. Equation (4) is equivalent to

$$\mathbf{x} = \begin{pmatrix} | & | & | \\ \mathbf{b_1} & \mathbf{b_2} & \mathbf{b_3} \\ | & | & | \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \equiv B\mathbf{y}$$

Therefore, $\mathbf{y} = B^{-1}\mathbf{x}$.

11. **Solving linear equations.**

• The equations can be rewritten in matrix form as

$$\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 20 \\ 4 \end{pmatrix}$$

The inverse of the 2x2 matrix is:

$$\frac{1}{5}\begin{pmatrix} 1 & 3 \\ 1 & -2 \end{pmatrix}$$

Therefore, x and y is given by:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 1 & 3 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 20 \\ 4 \end{pmatrix} = \begin{pmatrix} 6.4 \\ 2.4 \end{pmatrix}$$

12. **Solving linear equations.**

• The equations are identical to the ones from Problem 11, just replace x with e_1 and y with e_2. Therefore $e_1 = 6.4$ and $e_2 = 2.4$.

13. **Index notation.**

• Let's write

$$A = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1N} \\ A_{21} & A_{22} & \dots & A_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ A_{N1} & A_{N2} & \dots & A_{NN} \end{pmatrix},\ B = \begin{pmatrix} B_{11} & B_{12} & \dots & B_{1N} \\ B_{21} & B_{22} & \dots & B_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ B_{N1} & B_{N2} & \dots & B_{NN} \end{pmatrix}$$

Then

$$AB = \begin{pmatrix} \sum\limits_i A_{1i}B_{i1} & \sum\limits_i A_{1i}B_{i2} & \dots & \sum\limits_i A_{1i}B_{iN} \\ \sum\limits_i A_{2i}B_{i1} & \sum\limits_i A_{2i}B_{i2} & \dots & \sum\limits_i A_{2i}B_{iN} \\ \vdots & \vdots & \ddots & \vdots \\ \sum\limits_i A_{Ni}B_{i1} & \sum\limits_i A_{Ni}B_{i2} & \dots & \sum\limits_i A_{Ni}B_{iN} \end{pmatrix}$$

and

$$BA = \begin{pmatrix} \sum\limits_i B_{1i}A_{i1} & \sum\limits_i B_{1i}A_{i2} & \dots & \sum\limits_i B_{1i}A_{iN} \\ \sum\limits_i B_{2i}A_{i1} & \sum\limits_i B_{2i}A_{i2} & \dots & \sum\limits_i B_{2i}A_{iN} \\ \vdots & \vdots & \ddots & \vdots \\ \sum\limits_i B_{Ni}A_{i1} & \sum\limits_i B_{Ni}A_{i2} & \dots & \sum\limits_i B_{Ni}A_{iN} \end{pmatrix}$$

The trace is the sum along the diagonal elements of a matrix. Therefore $\mathrm{Tr}(AB) = \sum\limits_j \sum\limits_i A_{ji}B_{ij}$ and $\mathrm{Tr}(BA) = \sum\limits_j \sum\limits_i B_{ji}A_{ij}$, and the two are equivalent.

• Take the trace of both sides of the equation, $\mathrm{Tr}(M) = \mathrm{Tr}(VDV^{-1})$. The generalization of the rule you learned in the previous part is that $\mathrm{Tr}(ABC...Z) = \mathrm{Tr}(BC..ZA) = \mathrm{Tr}(C..ZAB) = ...$ Therefore we can write $\mathrm{Tr}(M) = \mathrm{Tr}(VDV^{-1}) = \mathrm{Tr}(DV^{-1}V) = \mathrm{Tr}(DI) = \mathrm{Tr}(D)$.

• This is easiest to see if the equation is rewritten as $MV = VD$. The rule for determinants is $\det(AB...Z) = \det(A)\det(B)...\det(Z)$. Therefore, taking the determinant: $\det(MV) = \det(VD) = \det(M)\det(V) = \det(V)\det(D)$, hence $\det(A) = \det(D)$.

14. **Rotation Matrices.**

• The dot product of the two rows is $-\cos\theta\sin\theta + \sin\theta\cos\theta = 0$.

• The dot product of the two columns is $\cos\theta\sin\theta - \sin\theta\cos\theta = 0$.

•

$$\begin{aligned} R^{\mathrm{T}}R &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

• $\mathbf{x}' = R\mathbf{x} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\theta + y\sin\theta \\ -x\sin\theta + y\cos\theta \end{pmatrix}$

The length of $\mathbf{x}'$ is

$$\begin{aligned} |\mathbf{x}'| =& x'^2 + y'^2 = (x\cos\theta + y\sin\theta)^2 + (-x\sin\theta + y\cos\theta)^2 \\ =& x^2\cos^2\theta + y^2\sin^2\theta + 2xy\cos\theta\sin\theta + x^2\sin^2\theta + y^2\cos^2\theta - 2xy\cos\theta\sin\theta \\ =& x^2(\cos^2\theta + \sin^2\theta) + y^2(\cos^2\theta + \sin^2\theta) = x^2 + y^2 = |\mathbf{x}| \end{aligned}$$

15. **Eigenvalues and eigenvectors**

• $A\mathbf{x} = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 2 \\ 1 \end{pmatrix}$. The eigenvalue is 3.

• To find the eigenvalues, we need to solve the characteristic equation

$$|A - \lambda I| = \begin{vmatrix} 4-\lambda & -2 \\ 1 & 1-\lambda \end{vmatrix} = (4-\lambda)(1-\lambda) + 2 = 0$$

Since we know that $\lambda = 3$ is one of the eigenvalues, we can write this as $(\lambda - 3)(\lambda - 2) = 0$ (alternatively, we can just solve for the quadratic equation). Therefore, the second eigenvalue is 2. To get the eigenvectors, we need to find the values x and y that satisfy

$$\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2 \begin{pmatrix} x \\ y \end{pmatrix}$$

Let's rewrite this as

$$\begin{pmatrix} 4x - 2y \\ x + y \end{pmatrix} - 2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x - 2y \\ x - y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Looking at either the top or bottom row shows that any vector where $x = y$ will be an eigenvector. Let's just use the value $(1\ 1)^{\mathrm{T}}$. Then

$$\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Therefore, $(1\ 1)^{\mathrm{T}}$ is indeed an eigenvector with eigenvalue 2.

16. **Eigenvalues and eigenvectors**

• The characteristic equation is

$$\begin{vmatrix} 3 - \lambda & 2 \\ 3 & -2 - \lambda \end{vmatrix} = (3 - \lambda)(-2 - \lambda) - 6 = (\lambda - 4)(\lambda + 3) = 0$$

Therefore, the two eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = -3$. To get the eigenvectors, we note

$$\begin{pmatrix} 3 & 2 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 4 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \text{ and } \begin{pmatrix} 3 & 2 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = -3 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

Working through the multiplication and subtracting right hand sides from left hand sides:

$$\begin{pmatrix} -x_1 + 2y_1 \\ 3x_1 - 6y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 6x_2 + 2y_2 \\ 3x_2 + y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Therefore, vectors satisfying the conditions $x_1 = 2y_1$ and $3x_2 = -y_2$ are eigenvectors, e.g. the vectors $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ are eigenvectors, with eigenvalues 4 and -3 respectively.

• The characteristic equation is $\lambda^2 - 2\lambda + 5 = 0$ whose solutions are $\lambda_\pm = 1 \pm 2i$ and whose eigenvectors satisfy $y_+ = ix_+$ and $y_- = -ix_-$.

• You can define the matrix in Matlab using $\mathrm{M=[2\ -3\ 1;\ 1\ -2\ 1;\ 1\ -3\ 2]}$; and obtain the eigenvalues and eigenvectors by calling $\mathrm{[V\ D]=eig(M)}$, where the columns of V are the eigenvectors and D is a diagonal matrix of eigenvalues. This should return

$$V = \begin{pmatrix} 0.8165 & -0.5774 & -0.6786 \\ 0.4082 & -0.5774 & 0.0186 \\ 0.4082 & -0.5774 & 0.7343 \end{pmatrix} \text{ and } D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

• The trace of a matrix M is given by $\sum_i M_{ii}$, which is equal to 1 for the first matrix, 2 for the second matrix, and 2 for the third matrix. In all three cases, the trace is equal to the sum of the eigenvalues. For the 2×2 matrices, the determinants are $-2 \cdot 3 - 2 \cdot 3 = -12$ and $1 \cdot 1 + 2 \cdot 2 = 5$, both of which are equal to their respective products of eigenvectors. The 3×3 matrix can be performed by hand using Laplace development, but its much simpler in Matlab to call $\mathrm{det(M)}$ which returns the value 0.

17. **Eigenvectors of symmetric matrices.** My apologies, because this problem is actually more difficult than I intended, and I realized after the tutorial that my method only shows that $V^{\mathrm{T}}V$ commutes with D. There is also one crucial piece of information that I neglected which makes this problem more manageable: assume the eigenvectors are distinct.

    Now imagine that the vectors $\mathbf{v_1}$ and $\mathbf{v_2}$ are both eigenvectors of the matrix A with distinct eigenvalues λ_1 and λ_2 respectively. If the dot product between $\mathbf{v_1}$ and $\mathbf{v_2}$ is zero, then the two are orthogonal. Since, $\mathbf{v_1}$ and $\mathbf{v_2}$ are eigenvectors, $A\mathbf{v_1} = \lambda_1\mathbf{v_1}$ and $A\mathbf{v_2} = \lambda_2\mathbf{v_2}$. That means that the dot product between $A\mathbf{v_1}$ and $\mathbf{v_2}$ is also equal to zero if both eigenvectors are parallel. Let's calculate the dot product between $A\mathbf{v_1}$ and $\mathbf{v_2}$:

$$\lambda_1\mathbf{v_1}^{\mathrm{T}}\mathbf{v_2} = (\mathbf{v_1}^{\mathrm{T}}A^{\mathrm{T}})\mathbf{v_2} = (\mathbf{v_1}^{\mathrm{T}}A)\mathbf{v_2} = \mathbf{v_1}^{\mathrm{T}}(A\mathbf{v_2}) = \lambda_2\mathbf{v_1}^{\mathrm{T}}\mathbf{v_2},$$

    where we have replaced A for A^{T} since A is symmetric. Therefore, $\lambda_1\mathbf{v_1}^{\mathrm{T}}\mathbf{v_2} = \lambda_2\mathbf{v_1}^{\mathrm{T}}\mathbf{v_2}$ and because we stipulated that $\lambda_1 \neq \lambda_2$, the condition can only be met if $\mathbf{v_1}^{\mathrm{T}}\mathbf{v_2} = 0$. Therefore, we have shown that any two eigenvectors with distinct eigenvalues, will be perpendicular. Now, if we choose eigenvectors that are normalized, i.e. $\mathbf{v_i}^{\mathrm{T}}\mathbf{v_i} = 1$, then we have proved that $V^{\mathrm{T}}V = I$ for symmetric matrices A where the eigenvectors are distinct.

18. **Eigenvectors of symmetric matrices.** Define the matrix in Matlab: $\mathrm{A=[9\ 3\ 6;\ 3\ 7\ 4;\ 6\ 4\ 3]}$; and get the eigenvectors $\mathrm{[V\ D]=eig(M)}$. The eigenvectors are in the columns of V. Calculate the dot product between any two eigenvectors using V(:,m)'*V(:,n) which will yield 0 when $\mathrm{n \neq m}$ and 1 when $\mathrm{n=m}$.