Time Series I — Linear Analysis

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Usually, it is not possible to measure all the dynamical variables of interest. For example, when conducting experiments on nerve cells, the transmembrane potential is measured, but not the recovery variables. Also the measurement will contain some error, and will thus be related to the actual voltage through some equation with noise.

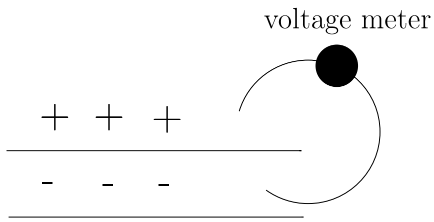


Figure 1: Transmembrane potential

Different types of noise

- measurement error — difference between measurement and actual value of variable
- systematic error — consistent error due to flaw in measurement process
- dynamical noise — noise affecting dynamical variables (e.g. due to influences not accounted for in model)

Given the existence of noise, what can we infer from a set of measurements about the underlying system?

Time-series analysis

Suppose we have a list of data points $x_1, x_2, \ldots, x_n$ taken at particular times. Are they governed by an ODE or FDE, or are they random?

Two quantities of interest are the sample mean and the sample variance:

sample mean $$m_N \equiv \frac{1}{N}\sum_{i=1}^{N} x_i$$ analogous to μ

sample variance $$\sigma_N^2 \equiv \frac{1}{N-1}\sum_{i=1}^{N}(x_i - m_N)^2$$ analogous to σ^2

The $N-1$ in the denominator of the sample variance assures that σ_N^2 when averaged over many data samples will approach the true variance σ^2.

m_N is the "best estimate" of the center of the data points:

$$\text{Error} = E = \sum_{i=1}^{N} (x_i - M_{\text{est}})^2$$
$$\frac{\mathrm{d}E}{\mathrm{d}M_{\text{est}}} = 2\sum_{i=1}^{N} (x_i - M_{\text{est}}) = 0,$$

so $M_{\text{est}} = m_N = \frac{1}{N}\sum_{i=1}^{N} x_i$.

m_N is an estimate of the actual mean μ of $x(t)$, and σ_N^2 is an estimate of the actual variance σ^2 of $x(t)$:

$$m_N = \mu \ + \ \text{uncertainty}$$
$$\sigma_N^2 = \sigma^2 + \ \text{uncertainty}.$$

Usually, one ignores the uncertainty in σ_N^2 and just uses $\sigma_N \approx \sigma$ as a measure of the spread of the data about the mean or sample mean. But uncertainty in $\sigma_N^2 \sim 1/\sqrt{N}$, so error could be significant for small sample size N.

Another quantity of interest is the standard error of the mean, which we have seen before:

$$\frac{\sigma_N}{\sqrt{N}},$$

which measures the uncertainty in estimating μ by m_N. Where does $\sigma_N/\sqrt{N}$ come from?

If the X_i's are random variables with mean μ_0 and variance σ_0^2, then as $N \to \infty$,

$$\sum_{i=1}^{N} X_i \sim N(\mu_0 N, \sigma_0\sqrt{N}),$$

so

$$m_N = \frac{1}{N}\sum_{i=1}^{N} X_i \sim N(\mu_0, \sigma_0/\sqrt{N}).$$

The sample mean m_N therefore has standard deviation $\sigma_0/\sqrt{N}$ about the true mean μ_0, that is, 68% of the time m_N is within $\sigma_0/\sqrt{N}$ of μ_0, and 95% of the time m_N is within $2\ \sigma_0/\sqrt{N}$ of μ_0.

So far we have not used the fact that the x_i are measurements in a time series. This is fine if we are making independent noisy measurements of a constant, but what if x is a dynamical quantity?

Example — Lotka-Volterra

$$\dot{x} = x(\alpha - \beta y) + \nu$$
$$\dot{y} = y(\gamma x - \delta) + \eta,$$

where the quantities ν and η are dynamical noise, e.g. noise due to temperature, humidity, other species, etc.

Also suppose there is uncertainty in measuring the prey population. We might model this measurement as

$$D(t) = x(t) + \xi(t),$$

where $\xi(t)$ is measurement noise.

Models

Following Kaplan and Glass, first consider linear FDEs:

$$\text{Model one:}\ \ D_t = x^* + W_t,$$

where W_t is white noise from measurement. The variable x has settled down to a fixed value x^* so that only variation is due to (white) measurement noise. So the D_t are uncorrelated in time and all we can do is measure sample mean and sample variable.

$$\begin{aligned} \text{Model two:}\ \ x_{t+1} &= A + \zeta x_t + \nu_t, \quad \text{with}\ \ |\zeta| < 1 \\ D_t &= x_t + W_t, \end{aligned}$$

where ν_t is dynamical noise and W_t is measurement noise. There are two distinct sources of noise: W_t only affects measured value at time t, but ν_t affects measured values at later times as well.

$$\begin{aligned} \text{Without noise:}\ x_t &= x^* + (x_0 - x^*)\zeta^t, \qquad x^* = \frac{A}{1-\zeta} \\ &= x^* + (x_0 - x^*)e^{t \ln \zeta} \qquad (0 < \zeta < 1). \end{aligned}$$

How can we deduce the exponential decay coefficient from the noisy data? First, subtract out the sample mean:

$$\tilde{V}_t = D_t - m_N,$$

and let

$$V_t = \tilde{V}_t - W_t = D_t - m_N - W_t = x_t - x^*,$$

then, assuming $m_N = \mu = x^*$, and starting with the equation for x_{t+1}, after a little algebra:

$$\tilde{V}_{t+1} = \zeta \tilde{V}_t + \nu_t + (W_{t+1} - \zeta W_t).$$

Then

$$V_{t+1} = \zeta V_t + \nu_t,$$

where ν_t is dynamical noise. Note that

$$V_t = x_t - x^*.$$

So the fluctuations about the mean satisfy a simple linear equation with dynamical noise. Of course, we measure $\tilde{V}_t$. How can we find ζ? Choose ζ_{est} that minimizes the difference between $\tilde{V}_{t+1}$ and $\zeta \tilde{V}_t$. That is, we want to minimize the error:

$$E = \sum_{t=1}^{N-1} (\tilde{V}_{t+1} - \zeta_{\text{est}} \tilde{V}_t)^2.$$

So set

$$\frac{\mathrm{d}E}{\mathrm{d}\zeta_{\mathrm{est}}} = \sum_{t=1}^{N-1} 2(\tilde{V}_{t+1} - \zeta_{\mathrm{est}}\tilde{V}_t)(-\tilde{V}_t) = 0.$$

So

$$\zeta_{\mathrm{est}} = \frac{\sum_{t=1}^{N-1} \tilde{V}_{t+1}\tilde{V}_t}{\sum_{t=1}^{N-1} \tilde{V}_t\tilde{V}_t}.$$

ζ_{est} is called the "correlation coefficient". If we have enough data points N, then $\zeta_{\mathrm{est}} \simeq \zeta$. A large ζ_{est} means that there is a high correlation in a scatter plot of $\tilde{V}_{t+1}$ vs. $\tilde{V}_t$, and thus a slow decay of fluctuations about the fixed point.

More generally, a useful measure of how V_{t+k} depends on V_k is

$$R_{\mathrm{norm}}(k) = \frac{\sum_{t=1}^{N-k} \tilde{V}_{t+k}\tilde{V}_t}{\sum_{t=1}^{N-k} \tilde{V}_t\tilde{V}_t}.$$

$R_{\mathrm{norm}}(k)$ is called the "normalized autocorrelation function". Note that $R_{\mathrm{norm}}(k = 1) = \zeta_{\mathrm{est}}$, $R_{\mathrm{norm}}(0) = 1$.

What could we deduce from $R(k)$ about the dynamics?

- Model one — Measurement noise only. There is a rapid fall off of $R(k)$, and ζ_{est} is small. The shape is often taken as the definition of white noise, but we'll see that the same $R(k)$ can arise from deterministic evolution.

- Model two — Includes dynamical noise. Why is $R(k)$ more slowly decaying in this case?

Autocorrelation function and power spectra

The autocorrelation function and the power spectrum are two equivalent, complementary ways of characterizing some time-dependent quantity.

Let $V(t)$ be a time-dependent variable. The autocorrelation function V is

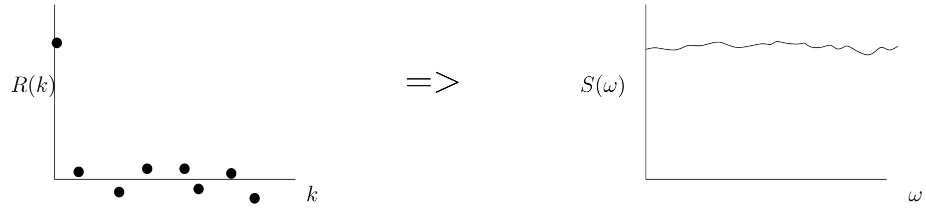
$$R(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} V(t)V(t+\tau)\,dt.$$

The power spectrum is the Fourier transform of $R(\tau)$:

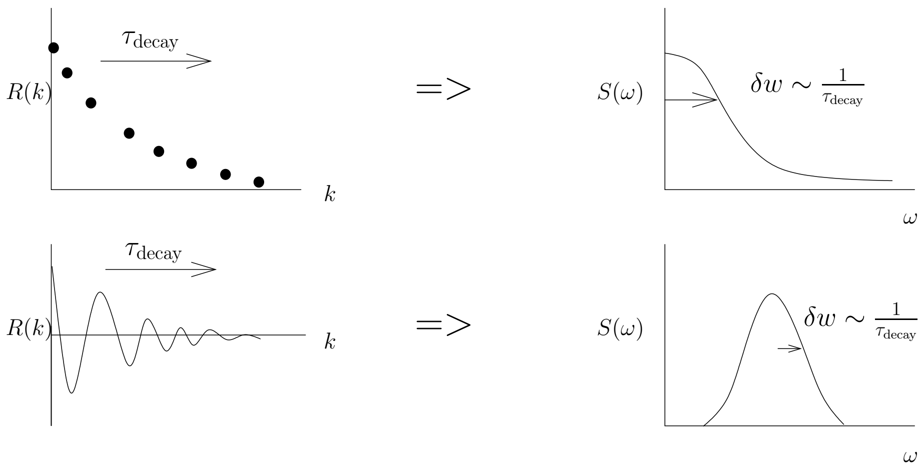
$$S(\omega) = \int_{-\infty}^{\infty} R(\tau)e^{-i2\pi\omega\tau}\,d\tau \geq 0.$$

The power spectrum gives the weight of the various frequency components ($\sin\omega t$ and $\cos\omega t$) in $V(t)$. What will $S(\omega)$ look like for the different $R(k)$ we have seen?

- Model one:



- Model two:


τ_{decay}
$R(k)$
k
=>
$S(\omega)$
$\delta w \sim \frac{1}{\tau_{\text{decay}}}$
ω
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k
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