Higher Dimensional Systems

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MOL 410/510

Higher dimensional linear systems

In general such systems are of the form:

$$\begin{aligned}
\frac{\mathrm{d}x_1}{\mathrm{d}t} &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1N}x_N \\
\frac{\mathrm{d}x_2}{\mathrm{d}t} &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2N}x_N \\
&\vdots \\
\frac{\mathrm{d}x_N}{\mathrm{d}t} &= a_{N1}x_1 + a_{N2}x_2 + \ldots + a_{NN}x_N,
\end{aligned}$$

so that

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = A\vec{x},$$

where A is the $N \times N$ matrix of coefficients a_{ij}. Note that one can write an N^{th} order differential equation in one variable x as N coupled 1^{st} order equations, for example by letting $x_1 = x, x_2 = \dot{x}$, etc.

Solutions can usually be written as

$$x_i(t) = \sum_j c_{ij} e^{\lambda_j t},$$

where the λ are the eigenvalues of A. As for 2-dimensional linear equations, eigenvalues are roots of characteristic equation:

$$\det\{\boldsymbol{A} - \lambda \boldsymbol{I}\} = 0,$$

an n^{th} order polynomial.

Eigenvalues

Eigenvalues λ are either real numbers or pairs of complex conjugates. Behavior as $t \to \infty$ is determined by eigenvalues:

- positive real parts $\Rightarrow$ divergence
- negative real parts $\Rightarrow$ exponential decay to 0
- complex conjugates $\Rightarrow$ oscillations (while diverging or decaying)
- zero real parts $\Rightarrow$ special cases, e.g. fixed lines, persistent oscillations.

Remark: Degenerate eigenvalues allow other solutions for $x_i(t)$. Take, for example, the damped harmonic oscillator:

$$\ddot{x} + 2\dot{x} + x = 0$$

Let $y = \dot{x}$, so that

$$\begin{aligned}\dot{x} &= \quad y \\ \dot{y} &= -x - 2y.\end{aligned}$$

Then

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$$

and

$$\begin{aligned}-\lambda(-2-\lambda) + 1 &= 0 \\ \lambda^2 + 2\lambda + 1 &= 0 \\ (\lambda+1)^2 &= 0 \\ \lambda_{1,2} &= -1,\end{aligned}$$

where the general solution is

$$x(t) = c_1 e^{-t} + c_2 t e^{-t}$$

Higher dimensional nonlinear systems

Example - Lorenz equations

The Lorenz equations are a simplified model of convective rolls in the atmosphere. In what follows, σ is the Prandtl number, r is the Rayleigh number, and b describes the aspect ratio. The Prandtl number σ, defined as ν/κ, where ν is the kinematic viscosity and κ is the thermal diffusivity, determines the type of convection. The Rayleigh number r, defined as $g\alpha\Delta T d^3/\nu\kappa$, determines the onset of convection.

These equations are:

$$\begin{aligned}\dot{x} &= \sigma(y - x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz\end{aligned}$$

Which are the nonlinear terms? The Lorenz system is *dissipative*, that is, volumes in phase space contract.

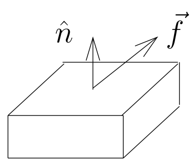

Figure 1: Phase Space

Consider the small volume in phase space shown in Fig. 1, where $\hat{n}$ is a surface normal and $\vec{f}$ is the instantaneous velocity in the phase space, i.e. $(\dot{x}, \dot{y}, \dot{z})$.

In a time $\mathrm{d}t$, a patch of area $\mathrm{d}A$ sweeps out a volume $(\vec{f} \cdot \hat{n})\,\mathrm{d}t\mathrm{d}A$. Therefore, we have

$$V(t + \mathrm{d}t) = V(t) + \int_S (\vec{f} \cdot \hat{n})\,\mathrm{d}t\mathrm{d}A,$$

so that

$$\begin{aligned} \dot{V} = \frac{V(t + \mathrm{d}t) - V(t)}{\mathrm{d}t} &= \int_S (\vec{f} \cdot \hat{n})\,\mathrm{d}A \\ &= \int_V (\vec{\nabla} \cdot \vec{f})\,\mathrm{d}V, \end{aligned}$$

where $(\vec{\nabla} \cdot \vec{f}) = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}$.

For the Lorenz system,

$$\begin{aligned} \vec{\nabla} \cdot \vec{f} &= \frac{\partial}{\partial x}[\sigma(y - x)] + \frac{\partial}{\partial y}[rx - y - xz] + \frac{\partial}{\partial z}[xy - bz] \\ &= -\sigma - 1 - b = -(\sigma + 1 + b), \end{aligned}$$

which is strictly less than 0. Therefore $V(t) = V(0)e^{-(\sigma+1+b)t}$, and volumes in phase space shrink exponentially.

So what can happen as $t \to \infty$?

- fixed point(s)
- limit cycle(s)
- ??

There are possible fixed points at $\dot{x} = \dot{y} = \dot{z} = 0$:

- The origin: $(x^*, y^*, z^*) = (0, 0, 0)$ Linear stability analysis:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix}.$$

  There is exponential decay in the z direction. For the 2 by 2 submatrix describing the x and y dynamics, the trace is $\tau = -\sigma - 1$ and the determinant is $\Delta = \sigma(1 - r)$, so the fixed point at the origin is stable for $r < 1$, unstable for $r > 1$.

- C^+ and C^-: $x^* = y^* = \pm\sqrt{b(r - 1)}$, $z^* = r - 1$. This is stable for $1 < r < r_H = \left(\frac{\sigma(\sigma+b+3)}{\sigma-b-1}\right)$.

  For $r > r_H$, there are no stable fixed points, no limit cycle, and no divergence: the system stays finite and collapses to a set of zero volume (by volume contraction), but it never settles down.

What does this mean? Chaos on a strange attractor.

What is chaos?

- Aperiodic behavior for all time,
- in a deterministic system,
- with sensitive dependence on initial conditions.

(Why couldn't we get chaos for $d = 2$? Because there is no crossing of trajectories.)

- "Aperiodic" means trajectories do not settle down to fixed points or periodic or quasi-periodic orbits as $t \to \infty$.
- "Deterministic" means that there are no random driving forces
- "Sensitive dependence on initial conditions" means that nearby trajectories separate exponentially fast.

How to quantify these notions?

Liapunov exponent

$\vec{x}(t)$ on the Lorenz attractor:

We start with $|\delta(0)| <<< 1$ (e.g. 10^{-15}), and then

$$|\vec{\delta}(t)| \sim |\vec{\delta}(0)|e^{\lambda t}$$

λ is the Liapunov exponent.

(As an aside, we need to average to get λ, and $|\vec{\delta}(t)|$ saturates. Note also that there are really N Liapunov exponents for an N dimensional system, $|\vec{\delta}(t)|$ is dominated by the largest.)

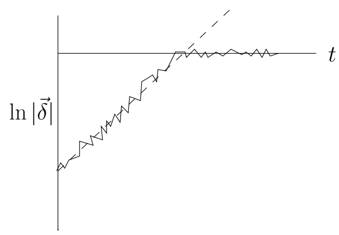

Figure 2: Liapunov exponent

$\lambda > 0$ supplies a "time horizon" for prediction.

$$|\vec{\delta}(t)| \sim |\vec{\delta}(0)|e^{\lambda t_{\text{horizon}}} \sim 1,$$

so that

$$t_{\text{horizon}} \sim \frac{1}{\lambda} \ln \frac{1}{|\vec{\delta}(0)|}$$

The time horizon therefore increases very, very slowly with the accuracy of initial conditions, which is why weather prediction is hard.

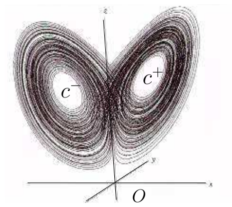

Figure 3: Lorenz System

The Lorenz map and finite-difference equations

Lorenz noted the following regarding the map:

> the trajectory apparently leaves one spiral only after exceeding some critical distance from the center. Moreover, the extent to which this distance is exceeded appears to determine the point at which the next spiral is entered; this in turn seems to determine the number of circuits to be executed before changing spirals again. It therefore seems that some single feature of a given circuit should predict the same feature of the following circuit.

Lorenz focused on z_n, the n^{th} local maximum of $z(t)$.

Specifically, he tried to see how z_n predicts z_{n+1}.

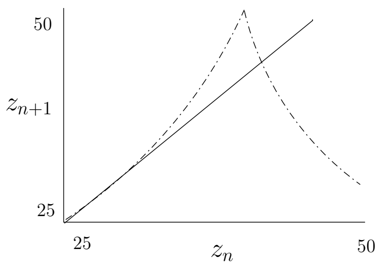

Figure 4: Lorenz Map

The points fall nearly on a curve! Just like finite difference equation: $|f'(z)| > 1$ for all z implies no stable fixed point or limit cycle!