Weakly Nonlinear Oscillators: "Cultural" Bonus
(no homeworks or exams on this topic)

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Introduction

We deal with equations of the form

$$\ddot{x} + x + \varepsilon h(x, \dot{x}) = 0,$$

where $0 < \varepsilon << 1$ and $h(x, \dot{x})$ is smooth.

Example — van der Pol oscillator

$$\ddot{x} + \varepsilon(x^2 - 1)\dot{x} + x = 0, \quad x(0) = 1, \dot{x}(0) = 0$$

for $0 < \varepsilon << 1$ is a weakly nonlinear oscillator.

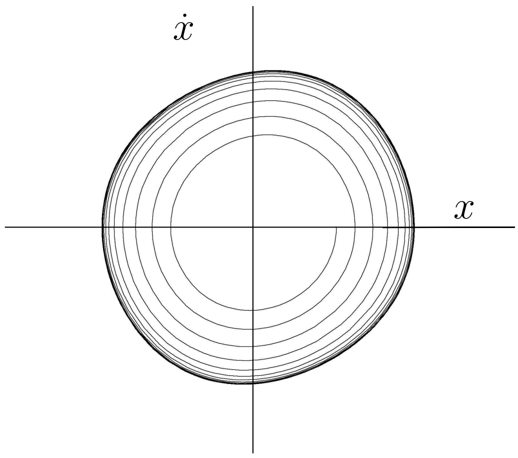

Figure 1: van der Pol oscillator with $\varepsilon = 0.1$.

This oscillator slowly approaches the limit cycle, a nearly circular orbit. Clearly, we have two time scales:

- single cycle $t \sim 1$
- change of amplitude $t \sim 1/\varepsilon$

We should be able to separate the time scales to approximately solve behavior of the van der Pol oscillator for $0 < \varepsilon << 1$, but we can't just expand the equation in ε.

What's wrong with simple ε expansion?

Consider the weakly damped linear oscillator

$$\ddot{x} + 2\varepsilon\dot{x} + x = 0, \quad x(0) = 0, \dot{x}(0) = 1.$$

We could try

$$x(t) = x_0(t) + \varepsilon x_1(t) + O(\varepsilon^2),$$

But, as a shortcut, since the oscillator is linear we can solve it exactly as an eigenvalue problem (i.e., $x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$):

$$x(t, \varepsilon) = (1 - \varepsilon^2)^{-1/2} e^{-\varepsilon t} \sin[(1 - \varepsilon^2)t].$$

If we solve by direct expansion ε, our solution will approximate

$$e^{-\varepsilon t} = 1 - \varepsilon t + O(\varepsilon^2).$$

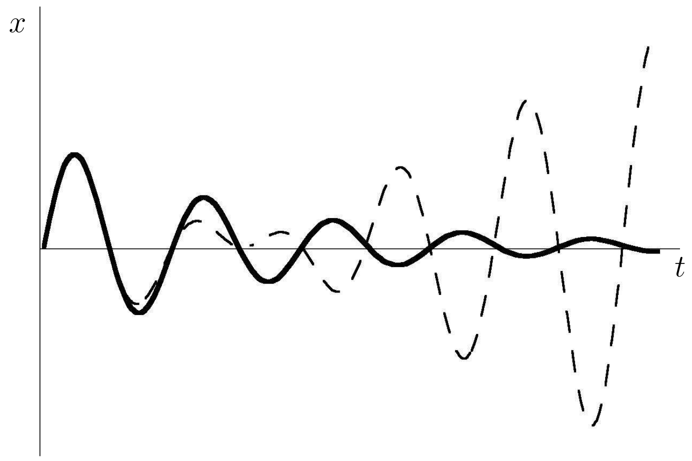

Figure 2: ε expansion. Solid curve - exact solution. Dashed curve - ε expansion.

This is only good for $\varepsilon t << 1$, and fails completely for $\varepsilon t > 1$! We need to take advantage of the two time scales.

Two-Timing

- Assume solution depends explicitly on two time scales: t, the regular time scale, and $T = \varepsilon t$, the long time scale, so we have $x(t, T)$. This makes sense in terms of van der Pol behavior: the amplitude depends on T, the phase of the orbit depends on t.
- Assume perturbation expansion

$$x(t, T) = x_0(t, T) + \varepsilon x_1(t, T) + O(\varepsilon^2).$$

Our approximate solution will be $x_0(t,T)$, and we will use the constraint that $x_1(t,T)$ must remain finite.

- Substitute expansion for x into the original equation. Use the chain rule for derivatives:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\partial x_0}{\partial t} + \frac{\partial x_0}{\partial T}\frac{\partial T}{\partial t} + \varepsilon(\frac{\partial x_1}{\partial t} + \frac{\partial x_1}{\partial T}\frac{\partial T}{\partial t}) + \ldots$$

Since $T = \varepsilon t, \frac{\partial T}{\partial t} = \varepsilon$. Therefore

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{\partial x_0}{\partial t} + \varepsilon(\frac{\partial x_0}{\partial T} + \frac{\partial x_1}{\partial t}) + O(\varepsilon^2) \\ \frac{\mathrm{d}^2x}{\mathrm{d}t^2} &= \frac{\partial^2 x_0}{\partial t^2} + \varepsilon(2\frac{\partial^2 x_0}{\partial T \partial t} + \frac{\partial^2 x_1}{\partial t^2}) + O(\varepsilon^2) \end{aligned}$$

van der Pol

We substitute this just-derived equation into the van der Pol equation

$$\ddot{x} + \varepsilon(x^2 - 1)\dot{x} + x = 0, \quad x(0) = 1, \dot{x}(0) = 0$$

and collect powers of ε. Since we can use any small number for ε, equations must be satisfied order by order in ε. We'll only work to 1st order, but we'll see that this will work quite well!

$$\begin{aligned} O(1) &: \frac{\partial^2 x_0}{\partial t^2} + x_0 = 0 \\ O(\varepsilon) &: 2\frac{\partial^2 x_0}{\partial T \partial t} + \frac{\partial^2 x_1}{\partial t^2} + (x_0^2 - 1)\frac{\partial x_0}{\partial t} + x_1 = 0 \end{aligned}$$

The $O(1)$ equation is a simple harmonic oscillator. So the most general solution is

$$x_0(t,T) = A(T)\cos(t) + B(T)\sin(t)$$

or, equivalently

$$x_0(t,T) = r(T)\cos(t + \phi(T)),$$

where r is the radius in the phase plane and $r(T)$ and $\phi(T)$ are slowly varying amplitude and phase of oscillation. This gives a whole family of solutions - which one is the right one? We substitute this solution for $x_0(t,T)$ into the $O(\varepsilon)$ equation, noting first that:

$$\begin{aligned} -(x_0^2 - 1)\ \frac{\partial x_0}{\partial t} &= [r^2\cos^2(t+\phi) - 1]r\sin(t+\phi) \\ -\ 2\ \frac{\partial^2 x_0}{\partial T \partial t} &= 2\frac{\partial}{\partial T}[r\sin(t+\phi)] \\ &= 2[r'\sin(t+\phi) + r\phi'\cos(t+\phi)]. \end{aligned}$$

We conclude that

$$\begin{aligned} \frac{\partial^2 x_1}{\partial t^2} + x_1 = {} & [r^2\cos^2(t+\phi) - 1]r\sin(t+\phi) \\ & + 2[r'\sin(t+\phi) + r\phi'\cos(t+\phi)] \end{aligned}$$

How does this define $r(T)$ and $\phi(T)$? The solution for x_1 must remain finite, so there can be no driving terms on the right-hand side $\sim \sin(t)$ or $\sim \cos(t)$ which will cause x_1 to diverge (i.e., no resonant driving on the right-hand side implies no secular terms $\sim t\sin t, t\cos t$ in x_1). So we expand the right-hand side:

$$\text{use } \cos^2(t+\phi)\sin(t+\phi) = \frac{1}{4}[\sin(t+\phi)+\sin 3(t+\phi)],$$

so

$$\begin{aligned}\frac{\partial^2 x_1}{\partial t^2} + x_1 = {} & [2r' - r + \frac{1}{4}r^3]\sin(t+\phi) \\ & + [2r\phi']\cos(t+\phi) \\ & + \frac{1}{4}r^3 \sin 3(t+\phi).\end{aligned}$$

So $2r' - r + \frac{1}{4}r^3 = 0$. Since $2r\phi' = 0$, we also have $\phi' = 0$. Recall the initial conditions $x(0) = 1, \dot{x}(0) = 0$. We have

$$\begin{aligned} r(0) &= \sqrt{x^2(0) + \dot{x}^2(0)} = 1 \\ \phi(0) &= \tan^{-1}\frac{\dot{x}(0)}{x(0)} \qquad = 0 \end{aligned}$$

Since $\phi' = 0$, $\phi(T) = 0$. So we need only solve

$$r' = \frac{\partial r}{\partial T} = \frac{1}{2}r - \frac{1}{8}r^3, \quad r(0) = 1$$

$$\begin{aligned}\int_0^T \mathrm{d}T &= \int_1^r \frac{\mathrm{d}r}{\frac{1}{2}r - \frac{1}{8}r^3} = \int_1^r \frac{8\mathrm{d}r}{r(4-r^2)} \\ T &= \int_1^r \frac{2\mathrm{d}r}{r} + \int_1^r \frac{2r\mathrm{d}r}{4-r^2} \\ T &= 2\log r\big|_1^r - \log(4-r^2)\big|_1^r \\ T &= 2\log r - \log(4-r^2) + \log 3 \\ T - \log 3 &= \log\frac{r^2}{4-r^2} \\ e^{T-\log 3} &= \frac{r^2}{4-r^2}\end{aligned}$$

$$\begin{aligned} 4\frac{e^T}{3} - r^2\frac{e^T}{3} - r^2 &= 0 \\ r^2(1 + \frac{1}{3}e^T) &= \frac{4}{3}e^T \\ r^2 = 4\frac{1/3e^T}{1 + 1/3e^T} &= \frac{4}{1+3e^{-T}} \\ r(T) &= \frac{2}{(1+3e^{-T})^{1/2}}.\end{aligned}$$

So

$$x(t,\varepsilon) = \frac{2}{(1+3e^{-T})^{1/2}}\cos t + O(\varepsilon),$$

where $T = \varepsilon t$. Note that as $T \to \infty$, $x(t,\varepsilon) \to 2\cos t$.

We have used our intuition that the solution to the van der Pol equation for $\varepsilon << 1$ has two distinct time scales to obtain an excellent approximate solution. This is a theme in applied math — find a method that implements "common sense."

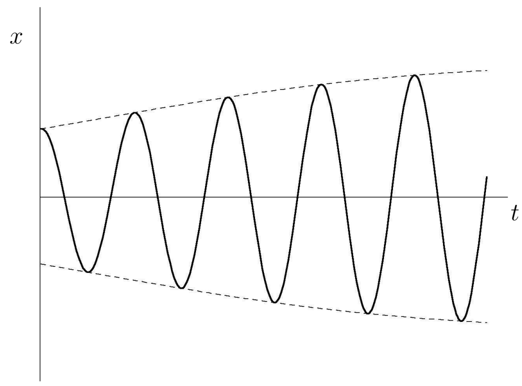

Figure 3: Two-timing approximation. Solid curve - exact solution. Dashed curve - two-timing approximate solution. The two curves are almost indistinguishable.