http://online.redwoods.cc.ca.us/instruct/darnold/laproj/fall98/kriscrg/fourier.pdf
http://www.annualreviews.org/doi/abs/10.1146/annurev.ne.08.030185.002555

I. MOTIVATING EXAMPLES

Let's think about signals through time.

Consider:

(1) A chemotaxing cell needs to distinguish intercellular signals from transient noise—which may have similar magnitude.

(2) Your eyes need to figure out where one thing starts and the next thing begins—they need to be good at picking out *edges*.

(3) Certain noises are much more interesting than others: even if the roar of a waterfall is louder than your friend's voice, you'd probably hear what your friend is saying better.

In each case, there are certain parts of a signal that we like, and certain parts we don't, and they're similar sizes

Just dropping everything below the "noise level" won't cut it

How do we get the good stuff?

Today we're going to talk about a class of information processing techniques known together as "Fourier analysis."

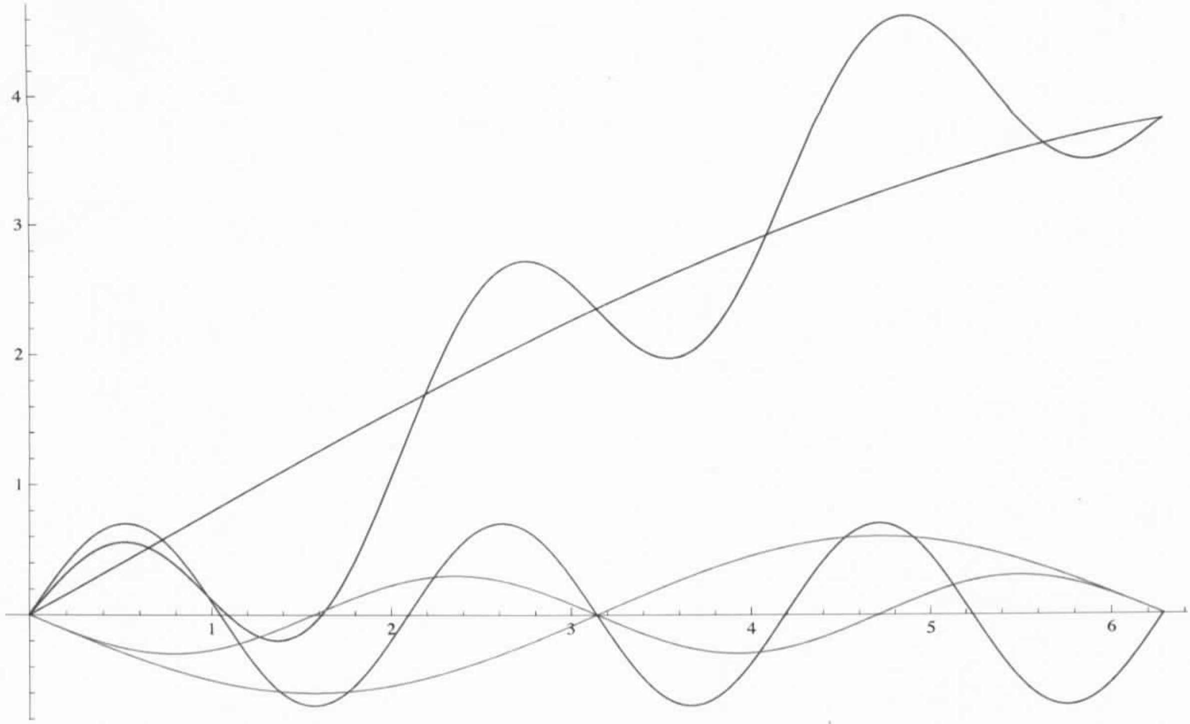
These techniques help in just these sorts of situations.

In[6]:= **f[x_] = 4 Sin[x / 5] - 0.6 Sin [x] - 0.3 Sin [2 x] + 0.7 Sin[3 x]**

Out[6]= $4\,\mathrm{Sin}\left[\frac{x}{5}\right] - 0.6\,\mathrm{Sin}[x] - 0.3\,\mathrm{Sin}[2\,x] + 0.7\,\mathrm{Sin}[3\,x]$

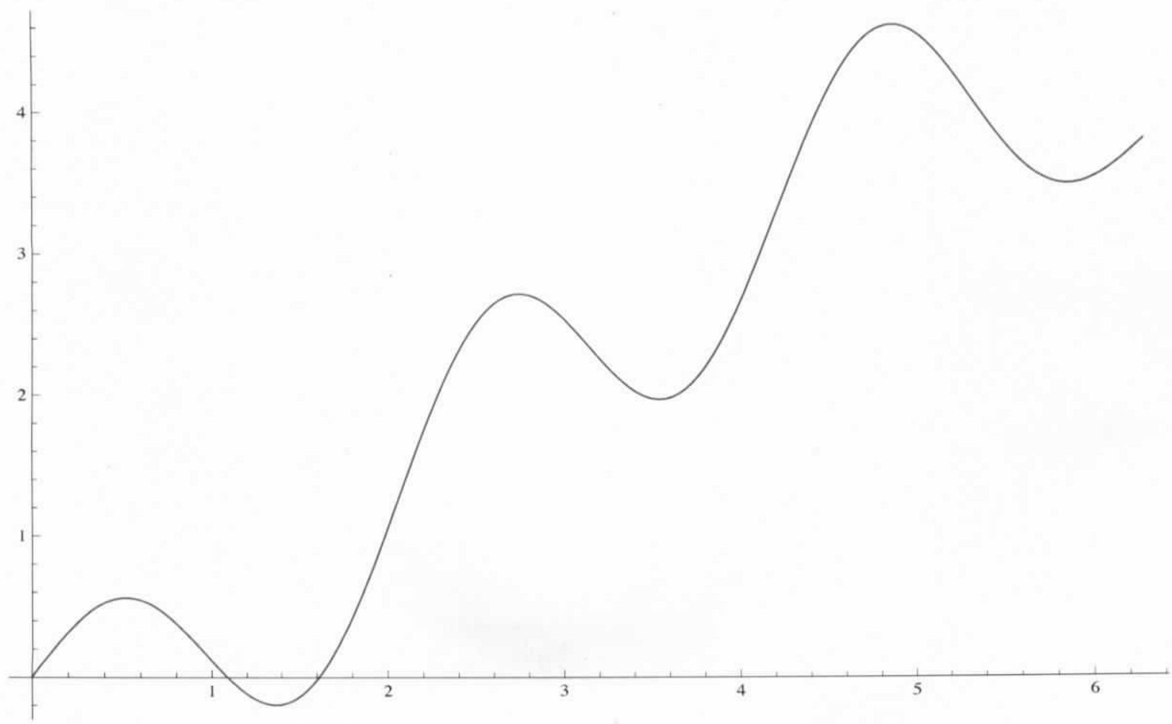
In[9]:= **Plot[{f[x], 4 Sin[x / 5], -0.6 Sin[x], -0.3 Sin[2 x], 0.7 Sin[3 x]}, {x, 0, 2 Pi}]**

Out[9]=



In[10]:= **Plot[f[x], {x, 0, 2 Pi}]**

Out[10]=



In general, we can go from thinking about how something changes over time to how it repeats periodically.

And this will potentially give a much clearer picture if the thing is periodic to begin with.

Our example function was ~~th~~ ~~prac~~ made out of sines

But we couldn't see that

In general, if we have a 2π-periodic f'n $f(x)$, we can write it as a sum of sines & cosines (Fourier <u>series</u>):

$$f(x) = \alpha_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

Where

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

Can also use a different 2π window as long as you're consistent

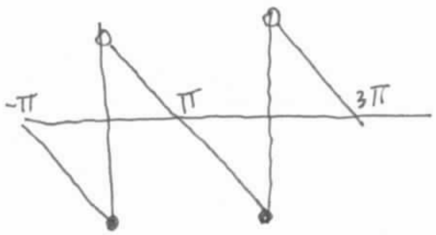
~~bu~~ We can do this with <u>any</u> 2π-periodic function.

Give an example, and then generalize to functions of any period like our motivating cases.

So to illustrate this works with ANY 2π-periodic function —

(Asmar 2000, p. 25)

$$f(x) = \begin{cases} \frac{1}{2}(\pi - x), & 0 < x < 2\pi \\ f(x+2\pi), & \text{else.} \end{cases}$$



note bounds — shifted to a better 2π region

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}(\pi - x)\,dx = \frac{1}{2\pi}\left[\frac{\pi x}{2} - \frac{x^2}{4}\right]_0^{2\pi} = 0$$

become sines

$$a_n = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{2}(\pi - x)\cos(nx)\,dx = \frac{1}{\pi}\int_0^{2\pi} \pi\cos(nx)\,dx - \int_0^{2\pi} x\cos(nx)\,dx = 0 - 0 = 0$$

Mathematica

$$b_n = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{2}(\pi - x)\sin(nx)\,dx = \frac{1}{\pi}\left[\frac{\sin(nx) - n(\pi - x)\cos(nx)}{2n^2}\right]_0^{2\pi}$$

Sin term is $\emptyset$ in both limits, this is

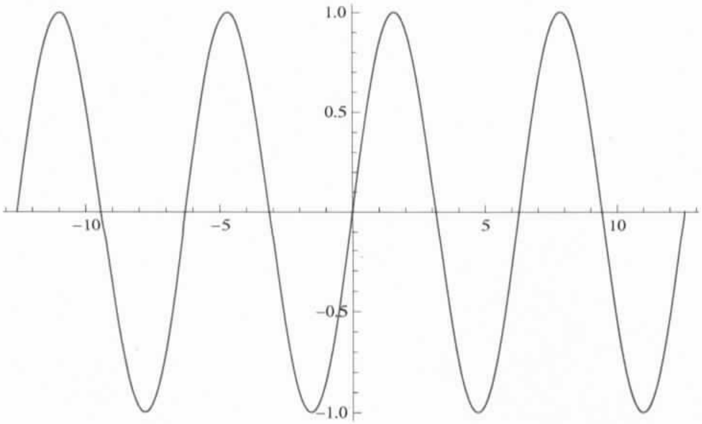
$$b_n = \frac{-\cancel{n}(\pi - x)\cos(nx)}{2\pi n^{\cancel{2}}}\Bigg|_0^{2\pi}$$

$$= \frac{-(\pi - 2\pi)\overbrace{\cancel{\cos(2\pi n)}}^{1}}{2\pi n} - \frac{(\pi - 0)\overbrace{\cancel{\cos(0)}}^{1}}{2\pi n} = \frac{1}{n}$$

So F.S. for sawtooth: $f(x) = \sum_{n=1}^{\infty} \frac{\sin(nx)}{n}$

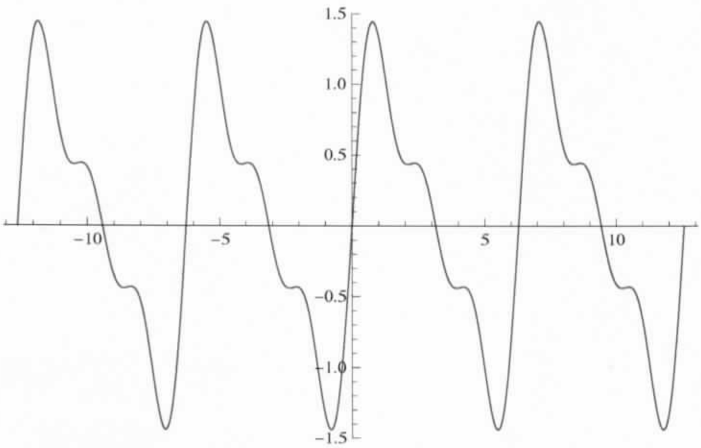
```
(* Sawtooth function -- Fourier series demonstration *)
Plot[Sin[x], {x, -4 Pi, 4 Pi}]
```

Out[43]=



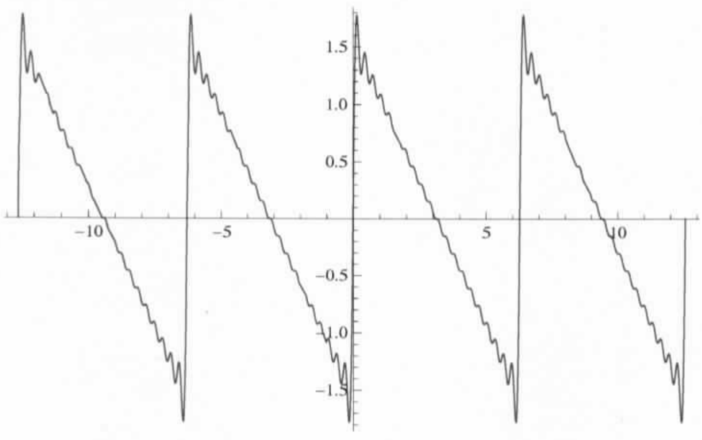
In[44]:= `Plot[Sin[x] + Sin[2 x] / 2 + Sin[3 x] / 3, {x, -4 Pi, 4 Pi}]`

Out[44]=



In[46]:= `Plot[Sum[Sin[n * x] / n, {n, 20}], {x, -4 Pi, 4 Pi}]`

Out[46]=



Discrete Fourier transform

So, who cares?

Bacterias, neurons, ears don't have access to Mathematica, and they're not going to do ~~any~~ algebra.

The idea there is that we want to ~~perform~~ ~~analysis~~ find out about the amplitude of responses with various periods

~~Then~~ Maybe that's all we want to do, or maybe we want to boost some & get rid of the others

Either way, this sort of analysis lets us separate lots of mixed signals into frequency vs. amplitude space.

At that point, we don't really care about the starting amplitude a_0 b/c we're imagining it went on forever.

We just want a continuous f/n of freq vs. amplitude

We call this the Fourier Transform

Going from time vs. amplitude to period vs amplitude

There's an algebraic way to do it

But mostly (in your life as well as in nature) it's done numerically.

Step 1: choose a period of time to watch. THIS IS YOUR MINIMUM OBSERVABLE FREQUENCY

Step 2: ~~break~~ divide into bins of Δt.

Step 3: look for repeating patterns from your bins. This is the FT.