MOL 410/510: Introduction to Biological Dynamics Fall 2012
Problem Set #7, Probability distributions and chaos (due 11/30/2012)
4 Questions, all are **MUST DO**

Clearly explain your answers, show all your work, and include any figures or code where requested. Don't forget that you can use Matlab's help function to get information on any of the functions introduced in this homework.

1. Consider a variable x that is described by a Gaussian distribution,
$$G(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right).$$
The mean value of x, called $\bar{x}$, is given by the integral
$$\mathrm{mean}(x) \equiv \bar{x} = \int_{-\infty}^{\infty} xG(x)dx.$$
The variance of x is given by the integral
$$\mathrm{var}(x) = \int_{-\infty}^{\infty} (x-\bar{x})^2 G(x)dx.$$
Evaluate the integrals to find the mean and the variance of x.

2. Let $f_1(x)$ be a Gaussian distribution with mean 0 and standard deviation σ_1, and let $f_2(x)$ be a Gaussian distribution with mean 0 and standard deviation σ_2. Compute
$$P(x) = \int_{-\infty}^{\infty} dx' f_1(x-x') f_2(x'). \tag{1}$$
This integral is called the **convolution** of $f_1(x)$ and $f_2(x)$. $P(x)$ describes the probability distribution of the sum of two independent random variables, one chosen from distribution f_1 and the other from distribution f_2. It can also be interpreted as the overlap of the probability distributions $f_1(x)$ and $f_2(x)$. The results of this problem are used in many places, for example in the computation of the standard deviation of the distribution of synaptic potentials generated from the summation of several quanta (del Castillo and Katz, 1954).

3. A series of action potentials in a nerve cell is described by a Poisson process. According to this, the probability that the time interval between an event and the second following event lies between t and $t + \Delta t$ is $p(t)\Delta t$, where

$$p(t) = R^2 t e^{-Rt}, \quad t \geq 0,$$

and R is a positive constant.

(a) Sketch $p(t)$ as a function of t. Show all maxima, minima, and inflection points.

(b) Evaluate $\int_0^\infty p(t)dt$. What is the interpretation of this integral?

(c) Evaluate $\int_0^\infty tp(t)dt$. What is the interpretation of this integral?

(d) Derive the formula given in the problem for the probability density $p(t)$ for the time interval between an event and the second following event in a Poisson process. You will need to use the fact that the probability distribution $p_1(t)$ for the time interval between an event and the next event is

$$p_1(t) = Re^{-Rt}, \quad t \geq 0.$$

(Hint: Apply the results of problem 2 above, since the time interval between an event and the second following event for a Poisson process can be viewed as a sum of two independent random variables, though in this case not Gaussian random variables.)

4. This problem will consider the so-called Lorenz attractor, or "strange attractor". Write a program to integrate the Lorenz oscillator equations

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz \end{aligned}$$

using the Euler method. Use the parameters $\sigma = 10$, $r = 28$, and $b = 8/3$.

(a) Start with a step size $dt = 0.05$ and make a plot of $x(t)$. Then reduce dt to 0.01, and see if $x(t)$ changes substantially. The changes become more dramatic as time increases. Repeat this process to find a satisfactory value of dt.

(b) Once you have settled on a value for dt, integrate the Lorenz equations starting from an initial condition that is very close to the attractor. You can find such an initial condition by starting at another arbitrary initial condition and integrating the equations until you are on the attractor. Then, pick off the last x, y, and z values to use as your new initial condition.

(c) Change the intial condition by a small amount, and see how long it takes for the sensitive dependence on initial conditions to create a very large change in $x(t)$ compared to that found in part (b). Do this again for other initial conditions that are even closer to that in part (b), and describe how the time that it takes for $x(t)$ to deviate dramatically from the $x(t)$ calculated in part (b) depends on the difference in initial conditions.

(d) Explain the meaning of the "Lyapunov exponent" for a strange attractor. How would you go about numerically estimating the Lyapunov exponent for the Lorenz attractor? Implement your method and report your estimate for the Lyapunov exponent.