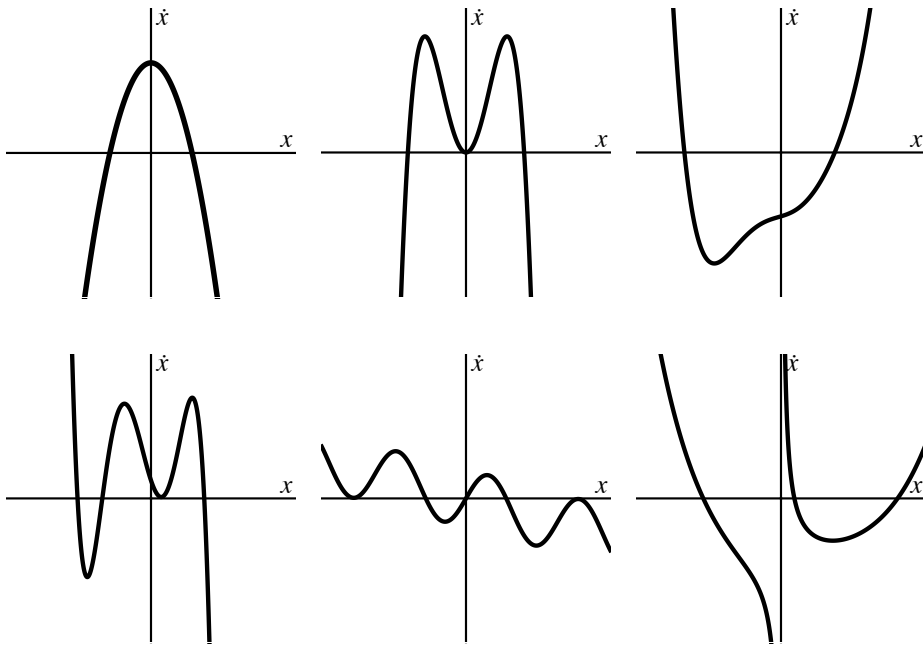


MOL 410/510: Introduction to Biological Dynamics Fall 2012
 Problem Set #4, Nonlinear Dynamical Systems (due 10/19/2012)
 6 **MUST DO** Questions, 1 **OPTIONAL** question

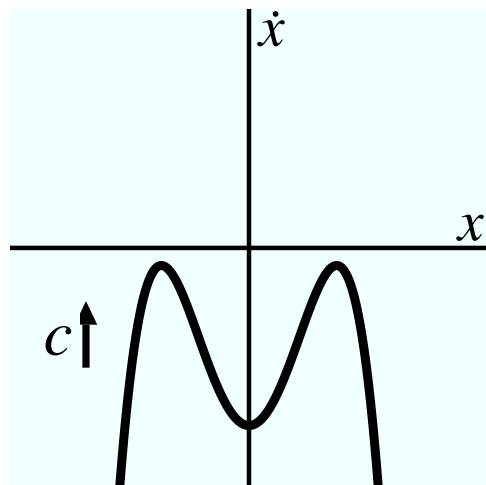
1. Below, several phase portraits are shown. For each curve:

- (a) (6 pts) Identify the fixed points.
- (b) (6 pts) Between each fixed point, draw an arrow that shows flow direction.
- (c) (6 pts) Determine the stability of each fixed point, i.e. stable, unstable, or saddle.

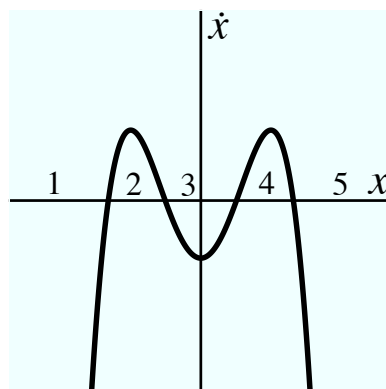
Assume that $\dot{x} = 0, x = 0$ where the axes meet.



2. (8 pts) Imagine the following phase portrait curve is parametrized by c , and as shown $c = 0$. Increasing the value of c moves the curve upward. By hand, draw an approximate bifurcation curve for this phase portrait as c increases. Exact positions of fixed points are unimportant. Instead, get the qualitative features of the curve correct. Make sure to indicate stable and unstable fixed points using solid and dashed lines respectively.

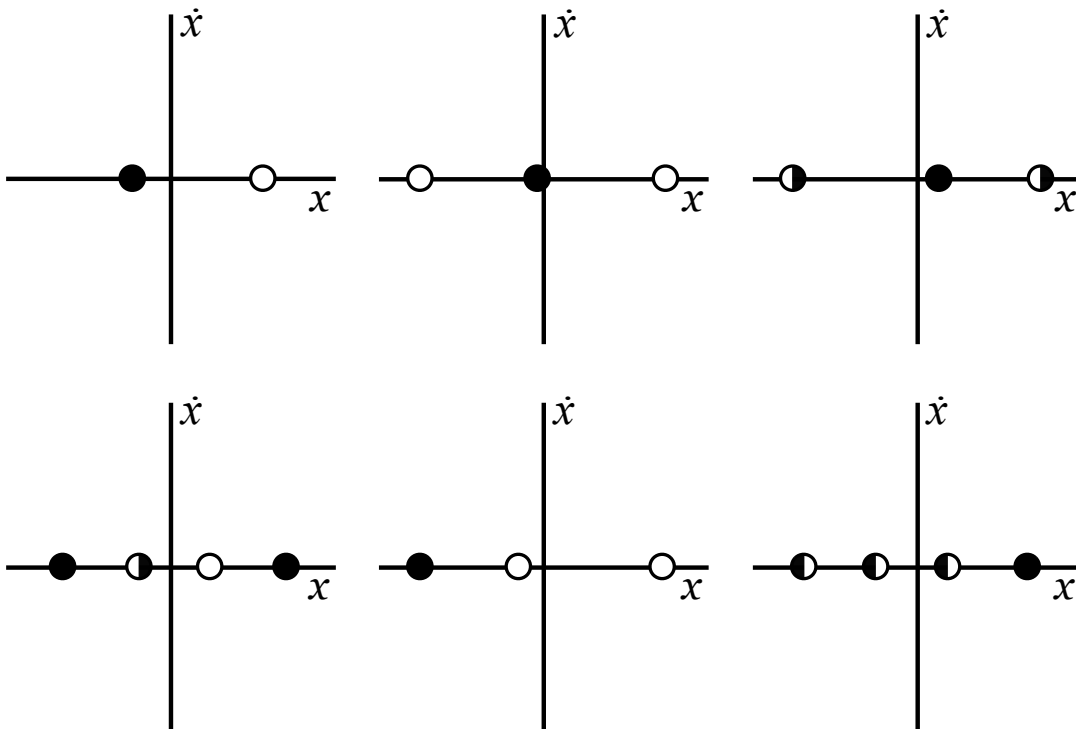


3. (10 pts) Imagine that a system follows the dynamics from the phase portrait shown below. Again by hand, draw a curve $x(t)$ that can approximate the dynamics of x ; that is, a curve that would be a solution to the differential equation that is described by the phase portrait. Use the points labeled (1,2,3,...) as $x(t = 0)$, and draw one curve for each initial condition.



4. Below, there are several examples of fixed points shown on an otherwise empty phase portrait. Closed circles represent stable fixed points, open circles represent unstable fixed points, and semi-open circles represent saddle points (where the closed side represents flow towards the stable point and the open side represents flow away from the stable point). **Assume there are no other fixed points than those that are shown.** Your task is to:

- (6 pts) Determine if the set of fixed points is possible given that **the phase portrait curve is continuous** and that **it is a function** (every value of x should have only a single value of \dot{x} associated with it. Do not worry about whether such a curve really exists, only if it is possible to cross or touch the zero line as suggested by the fixed points).
- (6 pts) If the set of fixed points is possible, draw the appropriate flow arrows onto the x -axis.
- (6 pts) Draw, by hand, a continuous function that matches each fixed point profile. Attempt to do this even for sets you found to be not possible. You should find that drawing a line with the necessary zero crossings/touchings is impossible.



5. In this problem, you will calculate the linear stability of a fixed point for the single neuron autapse circuit. The equation can be written as

$$\tau \frac{dx}{dt} = -x + w \tanh(x) \quad (1)$$

For simplicity, set $\tau = 1$, but do not assume any particular value for w .

- (a) (2 pts) First, let's plot the phase portrait of the system. Plot \dot{x} vs. x when $w = 0.5$, $w = 1$, and $w = 2$. Identify the fixed points and determine their stability in each case.
- (b) (2 pts) Recall that the Taylor series expansion for a function about the point x_0 is:

$$f(x) = f(x_0) + (x - x_0) \left. \frac{df(x)}{dx} \right|_{x_0} + \frac{(x - x_0)^2}{2!} \left. \frac{d^2 f(x)}{dx^2} \right|_{x_0} + \dots$$

Linearize equation (1) around the fixed point $x^* = 0$ up to first order (that is, take the Taylor series of (1) up to the first derivative). The first derivative of \tanh is given by:

$$\frac{d}{dx} \tanh x = 1 - \tanh^2 x \quad (2)$$

- (c) (2 pts) For what values of the synaptic weight w will the fixed point at $x^* = 0$ be linearly stable?
- (d) (2 pts) What is the Euler update equation for this system?
- (e) Simulate the dynamics of this system; i.e. use the Euler update equation you came up with earlier to numerically obtain a solution for the system. For all simulations set $\delta t = 0.01$, $\tau = 1$, and run the simulation from $t = 0$ s to $t = 10$ s. . Submit your code and a figure showing $x(t)$ vs t for each of the following sets of parameters:
- i. (2 pts) $x_0 = 0.5$ and $w = 0.25$
 - ii. (2 pts) $x_0 = 0.5$ and $w = 1$
 - iii. (2 pt) What is the difference between the dynamics and each case?
 - iv. (2 pts) Look at your phase portraits for $w = 0.25$ and $w = 1$. From them, argue why you see different dynamics in the two cases.
- (f) Now add a term for noise in your dynamics; that is, at every time step add $\sigma \sqrt{\delta t} \text{randn}$ to the value of x ; this will generate noise with mean 0 and std. dev. σ . Set $w = 2$ and $x_0 = 0$. Run the simulation with the following parameters, submit code and figures.
- i. (2 pts) set $\sigma = 0$.
 - ii. (3 pts) set $\sigma = 0.1$ and run the simulation for ten different trials. (You can put all ten trials on the same figure). Did you get the same result for all ten trials?

6. In this question you will examine a 2D circuit. Include all code and figures. The equations are

$$\frac{dx}{dt} = -x - w \tanh(y) \quad (3)$$

$$\frac{dy}{dt} = -y - w \tanh(x) \quad (4)$$

with $w > 0$.

- (a) (5 pts) What are the equations for the nullclines of the system? Using MATLAB, draw the two nullclines when $w = 0.5$. Now, do it for when $w = 2$. How many fixed points did you find in each case? Can you work out the stability of each point? What do all of the fixed points represent in terms of the steady-state of the network?
- (b) (5 pts) There is a fixed point at $x^* = 0, y^* = 0$. Let us find the conditions on w for when this fixed point is linearly stable. We will use a Taylor series expansion again, except now for two variables. The Taylor series expansion of a 2D function about the point x_0, y_0 is given by

$$\begin{aligned} f(x, y) = & f(x_0, y_0) + (x - x_0) \left. \frac{\partial f(x, y)}{\partial x} \right|_{x_0, y_0} + (y - y_0) \left. \frac{\partial f(x, y)}{\partial y} \right|_{x_0, y_0} \\ & + \frac{(x - x_0)^2}{2!} \left. \frac{\partial^2 f(x, y)}{\partial x^2} \right|_{x_0, y_0} + \frac{(y - y_0)^2}{2!} \left. \frac{\partial^2 f(x, y)}{\partial y^2} \right|_{x_0, y_0} \\ & + \frac{(x - x_0)(y - y_0)}{2!} \left. \frac{\partial^2 f(x, y)}{\partial x \partial y} \right|_{x_0, y_0} + \dots \end{aligned}$$

Linearize equations (3) and (4) about $x^* = 0, y^* = 0$, by performing the Taylor series expansion up to first order derivatives.

- (c) (5 pts) Write your equations in matrix form. Now, find the eigenvalues and eigenvectors of your matrix as a function of w . For what values of w is the fixed point at $x^* = 0, y^* = 0$ linearly stable? What type of fixed point does $x^* = 0, y^* = 0$ become when it does not meet this condition?
- (d) (5 pts) Use the Euler method to simulate the dynamics of this system. The update equations are given by

$$\begin{aligned} x_{n+1} &= x_n + \delta t(-x_n - w \tanh(y_n)) + \eta_x \\ y_{n+1} &= y_n + \delta t(-y_n - w \tanh(x_n)) + \eta_y \end{aligned}$$

where η_x and η_y are noise values independently sampled from a normal distribution with zero mean and standard deviation $\sigma\sqrt{\delta t}$ (we have set $\tau = 1$). Use $\delta t = 0.01$, $x_0 = 0$ and $y_0 = 0$, $w = 0.5$, $\sigma = 0.1$, and run the simulation up to $t = 10$. Repeat this simulation for ten different trials (don't repeat the noise in each trial but start from the same initial condition). Plot the dynamics of each variable for each trial.

- (e) (5 pts) Try the simulation with $w = 2$. First, do one simulation with no noise, i.e. $\sigma = 0$. What happens? Now set $\sigma = 0.1$. What happens after you put in the noise? Again, run the simulation for 10 trials. Do you get the same result on every trial?
- (f) **(OPTIONAL)** Animate your simulations with the phase portrait and nullclines. We will go through this step-by-step.
- i. To plot the phase portrait, we are going to use the MATLAB function `quiver`. To draw a field of arrows use `quiver(x,y,u,v)`. Where x,y are vectors representing the x - and y -positions of the tails of each arrow, and u,v are the x - and y -components of each arrow respectively. For example, try `quiver(0,0,1,0)` and `quiver([0 0],[0 0],[1 0],[0 1])`. `quiver` will scale the size of the arrows, so it will not always respect the absolute numbers in u and v . To get the vectors x and y , we need to make a grid. We can use the MATLAB function `meshgrid`. Try `[xgrid,ygrid]=meshgrid(-3:0.25:3,-3:0.25:3)`, then see what it does by typing `plot(xgrid,ygrid,'k')`. Evaluate the x - and y - components of each arrow by calculating \dot{x} and \dot{y} , from equations (3) and (4), at every point in `xgrid` and `ygrid`. Call `quiver`, now you have the phase portrait.
 - ii. On the same figure, we are going to plot the nullclines—lines for which the derivatives in (3) and (4) equal 0. Make an independent vector `z=linspace(-3,3,100)`. To get the nullcline for the y -direction, set $\dot{y} = 0$ to obtain the equation $y = -w \tanh x$. You can draw this line on the same graph with your arrows by typing `line(z,-w*tanh(z),'color','k')`. For the x -direction, set $\dot{x} = 0$ to obtain the equation $x = -w \tanh y$. Again, draw this line on the same graph using `line(-w*tanh(z),z,'color','r')` (notice the switch in arguments).
 - iii. Draw a single dot representing the initial state of the system, and return its handle. `h=line(x0,y0,'color','g','marker','!','markersize',28)`, where x_0,y_0 represent your initial x - and y -values. Now, iterate over all of your time steps, and update the position of your green dot using the `set` function. That is, at every time point call `set(h,'xdata',x(t),'ydata',y(t)); drawnow;` Do this for all 10 trials you simulated.

7. **(OPTIONAL)** Now, let's generalize our network to allow for all different kinds of connections. We are going to allow the neurons to have reciprocal connections and autapses, we are not going to make all synaptic weights the same value, and we will allow them to be negative or positive. We can write this system as

$$\frac{dx}{dt} = -x + w_{xx} \tanh(x) + w_{xy} \tanh(y) \quad (5)$$

$$\frac{dy}{dt} = -y + w_{yy} \tanh(y) + w_{yx} \tanh(x) \quad (6)$$

where w_{xy} indicates the synaptic strength that x receives from y , w_{yx} is the opposite, and w_{xx} and w_{yy} represent autapses. Simulate the equations (5) and (6) using the Euler method as detailed in the previous question. Do it for the following set of synaptic weights

- (a) $w_{xx} = w_{yy} = -w_{xy} = w_{yx} = 2$.
- (b) $w_{xx} = w_{yy} = w_{xy} = -w_{yx} = 2$.
- (c) $w_{xx} = w_{yy} = 2, w_{xy} = -1, w_{yx} = 2$.
- (d) $w_{xx} = w_{yy} = 2, w_{xy} = -0.5, w_{yx} = 2$.
- (e) $w_{xx} = w_{yy} = 2, w_{xy} = -w_{yx} = -0.1$.

For each set of weights, animate your simulations on top of quiver plots, but do not worry about plotting nullclines. (Hint, if you have not done it yet, “functionalizing” your code will help you perform the different parts with minimal extra work.)