**MOL 410/510: Introduction to Biological Dynamics** Fall 2012

Problem Set #3, Linear Dynamical Systems (due Friday 10/12/2012) 4 Questions, all are **MUST DO**

Clearly explain your answers, show all your work, and include any figures or code where requested. Don't forget that you can use Matlab's $\mathrm{help}$ function to get information on any of the functions introduced in this homework.

1. Suppose that the birth rate in the country Brobdingnag is proportional to the current population, with some fixed proportionality constant $b$ (in units of year$^{-1}$). And suppose that the death rate is also proportional to the current population, with constant $d$.

   (a) (5 pts) Write down the differential equation for the overall rate of change of the population $p(t)$, i.e., what is $\dot{p}(t)$?

   (b) (5 pts) What are the conditions (i.e., relative values of $b$ and $d$) under which a population of zero individuals would be a stable point for Brobdingnag? What are the conditions under which zero individuals is an unstable point?

   (c) (5 pts) If $b = 0.2$ and $d = 0.1$, how long does it take for the population to double?

   It turns out that Brobdingnagians are choosing to have very few children these days, with the result that although $d$ remains at $d = 0.1$, the value of $b$ has fallen to $b = 0.05$. In response, the Brodingnaginian king has started encouraging immigration, which comes in at a constant rate of $m$ people per year.

   (d) (5 pts) What is the differential equation for $p(t)$ now? What is the value of $m$ such that the Brobdingnagian population would stabilize at 20 million people?

   (e) (15 pts) Use Euler integration (namely, $p(t + \delta t) \approx p(t) + \dot{p}(t)\delta t$ ) in Matlab to numerically obtain the growth in population for $b = 0.05, d = 0.1, m = 2$ million, starting from $p(0) = 1$ million. Solve the same equation analytically, and plot the two solutions together. Start with a timestep size $\delta t = 0.1$ years. Now change the step size. What happens when $\delta t > 20$ years? How about $\delta t > 40$ years?

2. **Oscillations**

   Consider the equation

$$\dot{x} = \lambda x \tag{1}$$

   Plot $x$ as a function of time for complex $\lambda$, for the following cases:

   (a) (3 pts) $\lambda$ is purely imaginary, its real component is zero.

   (b) (3 pts) $\lambda$ has positive real component, and negative imaginary component.

   (c) (3 pts) $\lambda$ has positive real component, and positive imaginary component.

   (d) (3 pts) $\lambda$ has negative real component, and negative imaginary component.

   (e) (3 pts) $\lambda$ has negative real component, and positive imaginary component.

3. **Diagonal multi-dimensional dynamics**

   Suppose you have a set of one-dimensional differential equations, $\dot{x}_i = \lambda_i x$. Write the set of $\lambda_i$ into a diagonal matrix,

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & ... \end{pmatrix}$$

   You can then think of your set of independent differential equations as a single vector differential equation,

$$\dot{\mathbf{x}} = D\mathbf{x}$$

   (a) (5 pts) Thinking in these mutlidimenstional terms, what are the conditions under which the multidimensional origin, $\mathbf{x} = 0$, is a stable point? What happens if some $\lambda_i$ are less than zero and others are greater than zero?

   (b) (10 pts) Now suppose that you are starting with a multidimensional differential equation,

$$\dot{\mathbf{x}} = A\mathbf{x} \tag{2}$$

   where the square matrix $A$ is not necessarily diagonal. Remember that you can diagonalize to find $A = VDV^{-1}$, or, equivalently, $V^{-1}AV = D$, where $D$ is a diagonal matrix. Consider the change of basis to $\mathbf{y} = V^{-1}\mathbf{x}$.

   Find the differential equation for $\mathbf{y}$, i.e., find an expression for $\dot{\mathbf{y}}$. What are the conditions for $\mathbf{y} = 0$ being a stable point? What are the conditions for $\mathbf{x} = 0$ being a stable point?

4. **Multidimensional linear dynamics.** Say that you have two dynamical systems influencing each other such that.

$$\begin{aligned}\dot{x} &= -1.25x + 1.3y, \\ \dot{y} &= 1.3x + 0.25y\end{aligned}$$

(a) (3 pts) Write the matrix equation for this system in the form $\dot{\mathbf{u}} = A\mathbf{u}$, where $\mathbf{u}^{\mathrm{T}} = \begin{pmatrix} x & y \end{pmatrix}$.

(b) (15 pts) In this part, we will use Matlab to visualize the effect of the matrix $A$ on a set of points in space.

i. Create a grid of equally spaced points using the Matlab function $\mathrm{meshgrid}$ (look back to the tutorial from the first homework if you need help with this).

ii. Use Matlab so that all of the points in space can be written as a $2 \times N$ matrix, where each column is a vector describing the x- and y-position and $N$ is the total number of points you created.

iii. Multiply this matrix by the matrix $A$ you found in the previous part of the problem, let's call this new vector $\mathrm{Xdot}$.

iv. What do the values in $\mathrm{Xdot}$ represent?

v. Visualize $\mathrm{Xdot}$ by using the Matlab function $\mathrm{quiver}$ which produces a field of arrows. For example, calling quiver([4 3 2 3],[0 -1 0 1],[0 -1 0 1],[-1 0 1 0]) will make four arrows with starting positions at (4,0), (3,-1), (2,0), (3,1) and that have components (0,-1), (-1,0), (0,1), (1,0) respectively. Use your grid positions for the starting points of your arrows and use $\mathrm{Xdot}$ for the components of your vectors.

(c) (7 pts) Diagonalize the matrix $A$. On top of the quiver plot you just produced, draw two lines, one parallel to each eigenvector.

(d) (5 pts) Change basis to $\mathbf{z} = V^{-1}\mathbf{u}$, where $V$ is the matrix of eigenvectors that you found. What are the dynamics in $\mathbf{z}$ like? Does $z_1$ depend on $z_2$ or vice versa?

(e) (5 pts) On a new figure, redo a quiver plot in the diagonal space, i.e. use the diagonal matrix $D$ as your transformation matrix instead of $A$.